Implicit automata in typed λ-calculi

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Structure meets Power Workshop, June 28th 2021

Simply typed functions on Church numerals

Church encodings of (unary) natural numbers:

- $\mathsf{Nat} = (o \to o) \to o \to o$

- $n \in \mathbb{N} \rightsquigarrow \overline{n} = \lambda f. \, \lambda x. \, f \, (\dots (f \, x) \dots) : \mathsf{Nat}$ with n times f

- all inhabitants of Nat are equal to some $\overline{n}$ up to $=_{\beta\eta}$

Theorem (Schwichtenberg 1975)

The functions $\mathbb{N} \to \mathbb{N}$ definable by simply-typed λ-terms of type $\mathsf{Nat} \to \mathsf{Nat}$ are the extended polynomials.

(generated by $0, 1, +, \times,$ id and `ifzero`*)*

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Theorem (Schwichtenberg 1975)

The functions $\mathbb{N} \to \mathbb{N}$ definable by simply-typed λ-terms of type Nat $\to$ Nat *are the extended polynomials.*

(generated by 0, 1, $+$, $\times$, id and ifzero)

Let's add a bit of (meta-level) polymorphism: $t = \text{Nat}[A] \to \text{Nat}$
where $\text{Nat}[A] = \text{Nat}[A/o] = (A \to A) \to A \to A$

Open question

Choose some simple type A and some term $t : \text{Nat}[A] \to \text{Nat}$.
What functions $\mathbb{N} \to \mathbb{N}$ can be defined this way?

Simply typed functions on Church-encoded strings

To gain more insight, let's *generalize*! $\mathsf{Nat} = \mathsf{Str}_{\{1\}}$

Church encodings of *strings* over alphabet $\Sigma = \{a, b\}$:

- $\mathsf{Str}_{\{a,b\}} = (o \to o) \to (o \to o) \to o \to o$
- $abb \in \{a, b\}^* \leadsto \overline{abb} = \lambda f_a.\ \lambda f_b.\ \lambda x.\ f_a\ (f_b\ (f_b\ x)) : \mathsf{Str}_\Sigma$

More generally $\mathsf{Str}_\Sigma = (o \to o) \to \ldots |\Sigma| \text{ times} \ldots \to (o \to o) \to o \to o$

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Choose some simple type A and some term $t : \mathsf{Str}_\Gamma[A] \to \mathsf{Str}_\Sigma$.
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An answer for <u>predicates</u> [Hillebrand & Kanellakis 1996]

A subset of Σ^* is decidable by some $t : \mathsf{Str}_\Sigma[A] \to \mathsf{Bool}$ if and only if it is a *regular language*.

Note: unary regular languages $\cong$ ultimately periodic subsets of $\mathbb{N}$
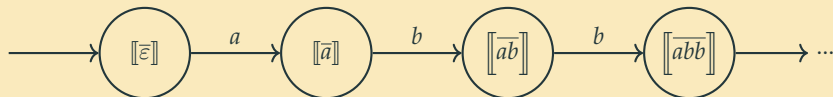
Theorem [Hillebrand & Kanellakis, LICS'96]

For any type A and any simply typed λ-term $t : \mathsf{Str}_\Sigma[A] \to \mathsf{Bool}$, $\{w \in \Sigma^* \mid t\,\overline{w} =_\beta \mathsf{true}\}$ is *regular*.

Proof by <u>semantic evaluation</u>.

Let $[\![-]\!]$ stand for a denotational semantics in the *CCC of finite sets*. (determined by $[\![o]\!]$)

We build an automaton with *finite* set of states $Q = [\![\mathsf{Str}_\Sigma[A]]\!]$
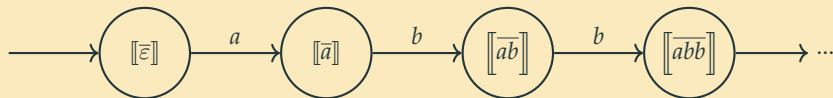
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$$t\,\overline{w} =_\beta \mathtt{true} \iff \llbracket t \rrbracket(\llbracket \overline{w} \rrbracket) = \llbracket \mathtt{true} \rrbracket \iff w \text{ accepted}$$
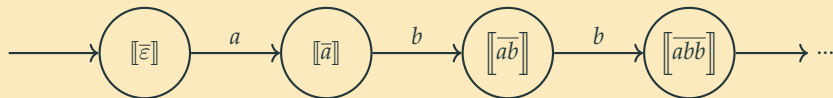
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To conclude: if $\mathrm{Card}([\![o]\!]) \geq 2$ then $[\![\mathsf{true}]\!] \neq [\![\mathsf{false}]\!])$ □
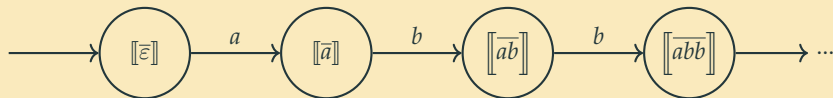

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Similar ideas in higher-order model checking (see e.g. Grellois & Melliès)

Regular functions

Assume a λ-calculus for linear intuitionistic logic with additives

- $\lambda^{\rightarrow} x.\, t : A \rightarrow B$ unrestricted function
- $\lambda^{\circ} x.\, t : A \multimap B$ linear function (exactly one x in t)
- coproducts $A \oplus B$ and products $A \,\&\, B$

Church encoding with linear types [Girard 1987]:

$$\overline{abb} = \lambda^{\rightarrow} f_a.\ \lambda^{\rightarrow} f_b.\ \lambda^{\circ} x.\, f_a\ (f_b\ (f_b\ x)) : \mathsf{Str}_{\{a,b\}} = (o \multimap o) \rightarrow (o \multimap o) \rightarrow o \multimap o$$

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Today's main theorem [Nguyễn & P.]

$$f : \Gamma^* \rightarrow \Sigma^* \text{ is a } \textit{regular function}$$
$$\Longleftrightarrow$$
f is defined by some $t : \mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$ in the intuitionistic linear λ-calculus
with A *purely linear*, i.e. containing no `$\rightarrow$'

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Regular functions are a classical topic, many equivalent definitions...
One of them: **copyless** *streaming string transducers* [Alur & Černý 2010]
$\rightsquigarrow$ sounds suspiciously like affine types!

Single-state streaming string transducers

Definition

- Finite set of Σ^*-valued *registers* e.g. $R = \{X, Y\}$

- Initial values $R \to \Sigma^*$ e.g. $X_{\text{init}} = Y_{\text{init}} = \varepsilon$

- *Register update function* e.g. $\quad a \mapsto \begin{cases} X := Xa \\ Y := aY \end{cases} \quad b \mapsto \begin{cases} X := Xb \\ Y := bY \end{cases} \quad c \mapsto \begin{cases} X := aba \\ Y := YabaX \end{cases}$

- "output function" e.g. out $= XY$

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Execution over *abaa*: start with

$$X = \varepsilon \qquad Y = \varepsilon$$

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$$X = a \qquad Y = a$$

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$$X = ab \qquad Y = ba$$

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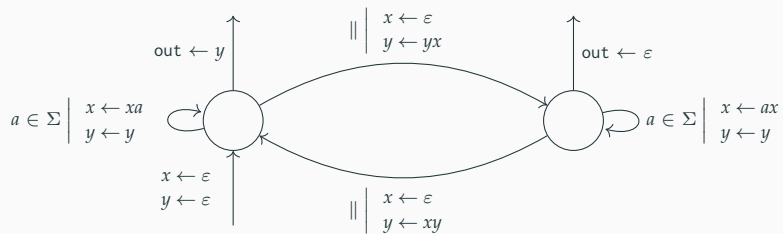- "output function" e.g. $\text{out} = XY$

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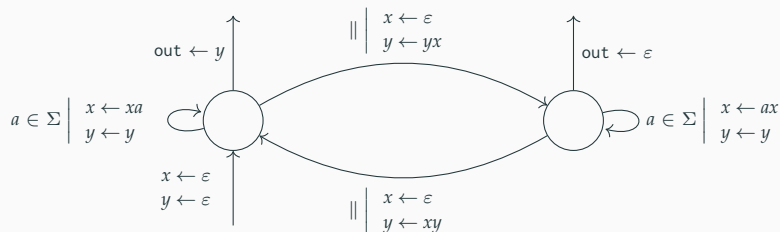*f* restricted to $\{a, b\}^*$: corresponds to $w \mapsto w \cdot \text{reverse}(w)$

SSTs can also have *states*: their memory is $Q \times (\Sigma^*)^R$ (with $|Q| < \infty$)

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Copylessness restriction

Each register appears *at most once* on RHS of $\leftarrow$

(for each fixed input letter, at most once among all the associated $\leftarrow$)

Intuition: memory $M = Q \otimes \Sigma^* \otimes \ldots \otimes \Sigma^*$, transitions $M \multimap M$

$(Q \cong 1 \oplus \ldots \oplus 1, \mathsf{concat} : \Sigma^* \otimes \Sigma^* \multimap \Sigma^*)$

Categorical automata

A framework for "single-pass" automata [Colcombet & Petrişan 2017]

- internal memory = object of a *category* $\mathcal{C}$
- transitions = morphisms (and [letter $\mapsto$ transition] = functor $\mathcal{T}_\Sigma \to \mathcal{C}$)

$$\mathcal{T}_\Sigma \;=\; \bullet \xrightarrow{\hspace{1.5em}} \bullet \overset{\overset{a \in \Sigma}{\curvearrowright}}{\xrightarrow{\hspace{1.5em}}} \bullet \qquad \longrightarrow \qquad \mathcal{C}$$

- DFA = automata over the category of finite sets
- Copyless SSTs $\approx$ start from a category $\mathcal{R}$ of copyless register updates
     + add states by *free finite coproduct completion* $(-)_\oplus$

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Formally

A streaming setting $\mathfrak{C}$ with output X is a tuple $(\mathcal{C}, \mathbb{T}, \bot\!\!\!\bot, out)$ with

- $\mathcal{C}$ a category

- $\mathbb{T}$ and $\bot\!\!\!\bot$ objects of $\mathcal{C}$

- $out : \mathrm{Hom}_{\mathcal{C}}(\mathbb{T}, \bot\!\!\!\bot) \to X$ a set-theoretic-map

Notion of $\mathfrak{C}$-automaton   (abusively called $\mathcal{C}$-automata in the sequel)

The register category with output alphabet Σ

- **Objects:** finite sets R, S think register variables
- **Morphisms:** $\text{Hom}_{\mathcal{R}}\,(R, S)$ = maps $S \to (R + \Sigma)^*$ corresponding to copyless register affectations
$$\textstyle\sum_{s \in S} |f(s)|_r \leq 1$$

- Monoidal with $\otimes = +$
- Free affine monoidal category over an object $\Sigma^* = \{\bullet\}$, morphisms $\varepsilon, a : \mathbf{I} \to \Sigma^*$ for $a \in \Sigma$ and $cat : \Sigma^* \otimes \Sigma^* \to \Sigma^*$
- For the streaming setting, take $\top = \mathbf{I} = 0$ and $\bot = \Sigma^* = \{\bullet\}$

SSTs as categorical automata

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Definition of the free finite coproduct completion $\mathcal{C}_\oplus$

- **Objects:** formal finite sums $\bigoplus_{u \in U} C_u$ of objects of $\mathcal{C}$

  formally pairs $(U, (C_u)_{u \in U})$, U a finite set, $C_u \in \mathcal{C}_0$

- **Morphisms:** $\mathrm{Hom}_{\mathcal{C}_\oplus}\left(\bigoplus_u C_u, \bigoplus_v D_v\right) = \prod_u \sum_v \mathrm{Hom}_{\mathcal{C}}(C_u, D_v)$

  $\cong \sum_f \prod_u \mathrm{Hom}_{\mathcal{C}}(C_u, D_{f(u)})$

- Morphisms $\bigoplus_{q \in Q} R \;\to\; \bigoplus_{q \in Q} R$ correspond to transitions in a SST
- Canonical embedding $\mathcal{C} \to \mathcal{C}_\oplus$ allows to lift streaming settings

Transductions definable in linear λ-calculus can be turned into automata over a category $\mathcal{L}$ of purely linear λ-terms (w/ const $f_c : o \multimap o$ for $c \in \Sigma$)

Claim

$\mathcal{L}$-automata compute the same string functions as λ-terms.

Proof: syntactic analysis of normal forms

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Proof strategy for linear λ-definable $\implies$ regular function

Define a *functor* $\mathcal{L} \to \mathcal{R}_\oplus$ preserving enough structure

Useful fact: there is a canonical functor from $\mathcal{L}$ to any *symmetric monoidal closed category* with (co)products

Unfortunately $\mathcal{R}_\oplus$ is **not** monoidal closed…

Toward a monoidal closed category

So far, we encountered:

- $\mathcal{L}$: category of purely linear λ-terms (w/ const $f_c : o \multimap o$ for $c \in \Sigma$)
- $\mathcal{R}$: category of finite sets of registers and copyless assignments
- $\mathcal{R}_\oplus$: free finite coproduct completion of the latter (add states)

Now consider:

- the free finite *product* completion: $\mathcal{C} \mapsto \mathcal{C}_\& = ((\mathcal{C}^{\mathrm{op}})_\oplus)^{\mathrm{op}}$

$$\textbf{Objects:} \text{ formal products } \underset{x}{\&} C_x$$

- the composite completion $\mathcal{C} \mapsto \mathcal{C}_\& \mapsto (\mathcal{C}_\&)_\oplus$

$$\textbf{Objects:} \text{ formal sums of products } \bigoplus_u \underset{x}{\&} C_{u,x}$$

similar to de Paiva's *Dialectica* categories **DC**, think $\exists u. \forall x. \varphi(u, x)$

Goals toward our main theorem

- Structure: $(\mathcal{R}_\&)_\oplus$ has finite products and is monoidal closed
- Conservativity: $(\mathcal{R}_\&)_\oplus$-automata and $\mathcal{R}_\oplus$-automata are equivalent

Structure (1): generic remarks $(\mathcal{C}_{\&})_{\oplus}$

Tensorial products can be lifted to the completions

- The new tensorial products satisfy the additional laws

$$A \otimes (B \& C) \equiv (A \otimes B) \& (A \otimes C) \qquad A \otimes (B \oplus C) \equiv (A \otimes B) \oplus (A \otimes C)$$

- In particular, $(\mathcal{C}_{\&})_{\oplus}$ has distributive cartesian products

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Lemma (folklore observation about dependent Dialectica categories?)

If $\mathcal{C}$ is symmetric monoidal and $(\mathcal{C}_\&)_\oplus$ has the internal homs $A \multimap B$
for all $A, B \in \mathcal{C}$, then $(\mathcal{C}_\&)_\oplus$ is symmetric monoidal closed.

$$\left(\bigoplus_{u \in U} \;\bigotimes_{x \in X_u} A_x \right) \multimap \left(\bigoplus_{v \in V} \;\bigotimes_{y \in Y_v} B_y \right) = \bigotimes_{u \in U} \bigoplus_{v \in V} \bigotimes_{y \in Y_v} \bigoplus_{x \in X_u} A_x \multimap B_y$$

Lemma

$\mathcal{R}_\oplus$ has the internal homs $A \multimap B$ for all $A, B \in \mathcal{R}$.

The construction appears in the original SST paper [Alur & Černý 2010]
without the categorical vocabulary.

$$\begin{cases} X := abXcY \\ Y := ba \end{cases} \quad \rightsquigarrow \quad \text{shape} \begin{cases} X := Z_1 X Z_2 Y \\ Y := Z_3 \end{cases} \quad + \quad \text{parameters } Z_1 = ab, \ldots$$

copyless SST $\implies$ finitely many shapes: use as states; registers for parameters

Lemma

$\mathcal{R}_\oplus$ *has the internal homs* $A \multimap B$ *for all* $A, B \in \mathcal{R}$.

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Conclusion

$(\mathcal{R}_\&)_\oplus$ is symmetric monoidal closed (and almost affine).

Conservativity

Lemma

$(\mathcal{C}_{\&})_{\oplus}$ *automata are equivalent to non-deterministic* $\mathcal{C}_{\oplus}$ *automata.*

A uniformization ($\sim$ determinization) theorem is enough to conclude

Conservativity

$(\mathcal{R}_{\&})_{\oplus}$-automata are equivalent to standard SSTs.

- Uniformization already known [Alur & Deshmuk 2011]
- Argument implicitly based on monoidal closure!
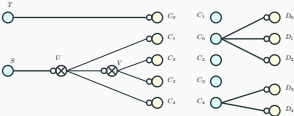
Theorem

For any monoidal category $\mathcal{C}$, if $\mathcal{C}_{\oplus}$ has all the internal homsets $A \multimap B$ for $A, B \in \mathcal{C}$, then $(\mathcal{C}_{\&})_{\oplus}$-automata and $\mathcal{C}_{\oplus}$-automata are equivalent.

equivalently: ND $\mathcal{C}_{\oplus}$-automata can be uniformized

Main results

I have just discussed

Today's main theorem [Nguyễn & P.]

$$\text{regular string function} \iff \begin{array}{l} \text{definable by some } t : \text{Str}_\Gamma[A] \multimap \text{Str}_\Sigma \\ \text{in ILL with } A \text{ purely linear} \end{array}$$

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Using similar tools, analogous result for trees over ranked alphabets

Main theorem for trees [Nguyễn & P.]

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Specific ingredients:

- Bottom-up categorical tree automata over SMCs
- A reasonably elegant multicategory of tree registers transition $\mathcal{R}$
  - Generated from the correponding PROP in a principled way   (reminiscent from the notion of clone)
  - Argument for $\mathcal{R}$-monoidal closure argument generalizes to trees
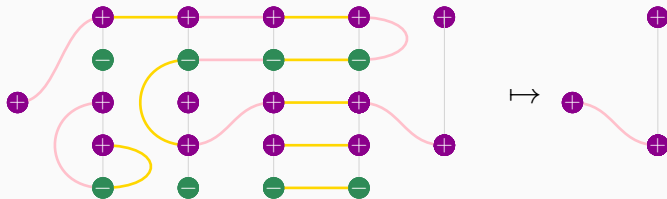- Regular functions already known to correspond to $\mathcal{R}_{\oplus\&}$-automata!

- Allows GoI-style interpretation in categories of diagrams $\qquad$ ($\cong$ **Int**(FinPartInj))
- $\rightsquigarrow$ Interpretation as two-way automata $\qquad$ [Hines 2003]

Define regular languages

Consequence (not interesting)

Every linear term $t : \mathsf{Str}_\Sigma[A] \multimap \mathsf{Bool}$ with $A \rightarrow$-free defines a regular language.
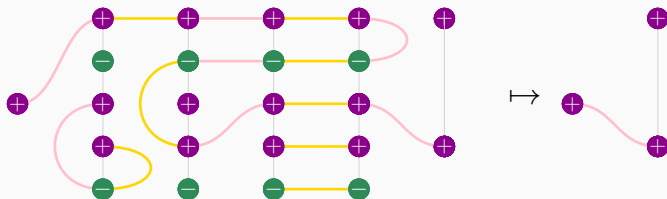
- Allows GoI-style interpretation in categories of **planar** diagrams
- ⤳ Interpretation as two-way **planar** automata

[Hines 2003,2006]

Define **star-free** languages

Consequence [Nguyễn, P. 2020]

Every **planar** linear term $t : \mathsf{Str}_\Sigma[A] \multimap \mathsf{Str}$ with $A \rightarrow$-free defines a star-free language.
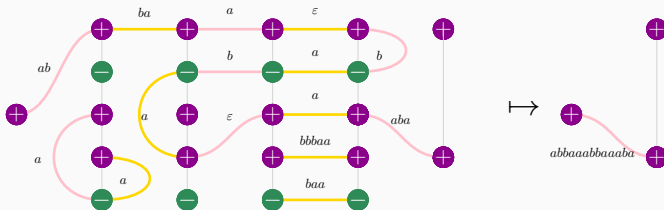
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Define **first-order** regular functions

Consequence

Every **planar** linear term $t : \mathrm{Str}_\Sigma[A] \multimap \mathrm{Str}$ with $A \to$-free defines a FO-transduction.
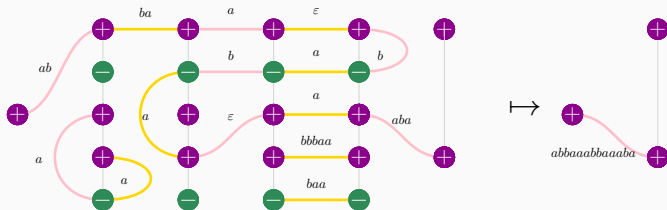
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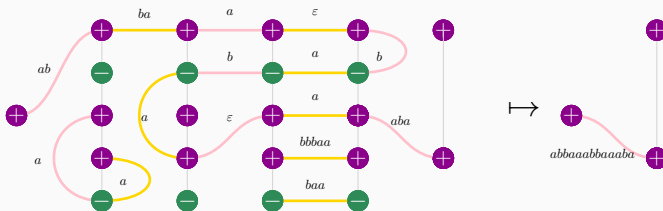Alas, planar linear terms are much weaker than FO-transductions   (preserve Parikh images)

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Conjecture

Every planar **affine** term $t : \mathsf{Str}_\Sigma[A] \multimap \mathsf{Str}$ with A →-free defines a FO-transduction.

The converse holds (main ingredient for the proof: the Krohn-Rhodes theorem)

Conclusion

What happened here:

- Connections between Church encodings and automata
- Application of categorical semantics (Dialectica, geometry of interaction (GoI))
- A generic uniformization-like construction $(\mathcal{C}_\&)_\oplus \to \mathcal{C}_\oplus$ for monoidal $\mathcal{C}$ with certain homsets

Some take-aways:

- Important ingredient in uniformization: monoidal closure

- Additive connectives are important for trees
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  - Further links with tree-walking automata?

Conclusion

Broader picture

$\mathsf{Str}_\Sigma[A] \multimap \mathsf{Bool}$ with A linear (adapted as needed):

λ-calculus	languages	status
simply typed	regular	✓ [Hillebrand & Kanellakis 1996]
linear or affine	regular	✓
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$\mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$ with A affine (adapted as needed):

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Thanks for listening! Questions?

- Interpret purely linear non-commutative λ-terms in a monoidal closed category
- We consider a non-commutative refinement of Geometry of Interaction

(well-known model of linear logic)

A compact closed category of planar diagrams

- **Objects:** words in $\{+, -\}^*$
- **Morphisms** $u \to v$: graphs over $|u| + |v|$ with
  - degree ≤ 1 for every node
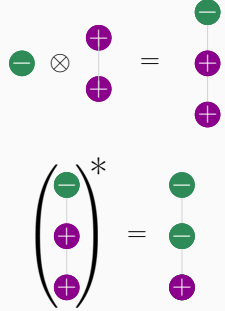 - polarity restrictions
  - planarity restriction
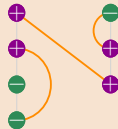
To compute the composition of two morphisms, follow the paths (and forget the middle component)
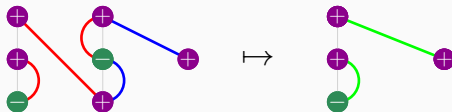
Structure to interpret the linear λ-calculus

- Monoidal product $A \otimes B$ given by concatenation
- Duals A^*: reverse and flip polarities
- Monoidal closure by setting $A \multimap B = A^* \otimes B$
- Interpretation of types $[\![A]\!]$ by induction with $[\![o]\!] = +$
  (injective interpretation of booleans)
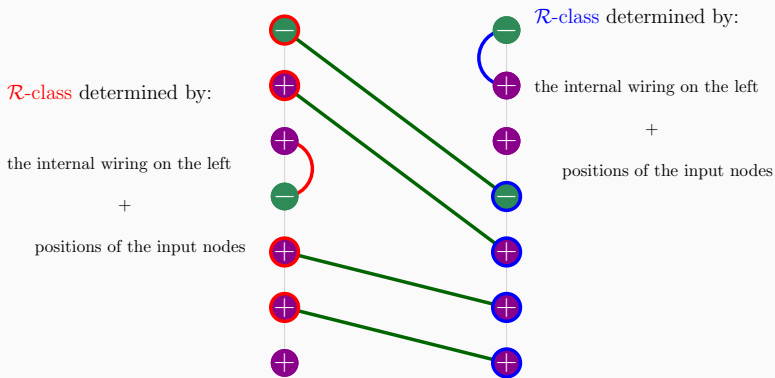

Examples

$$[\![((o \multimap o) \multimap o \multimap o) \multimap ((o \multimap o) \multimap o) \multimap o]\!] = -++---++$$

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To conclude, we need to show that every $(\mathsf{Hom}(A, A), \circ)$ is finite and aperiodic for every A

$\mathcal{R}$-class determined by:

the internal wiring on the left

+

positions of the input nodes

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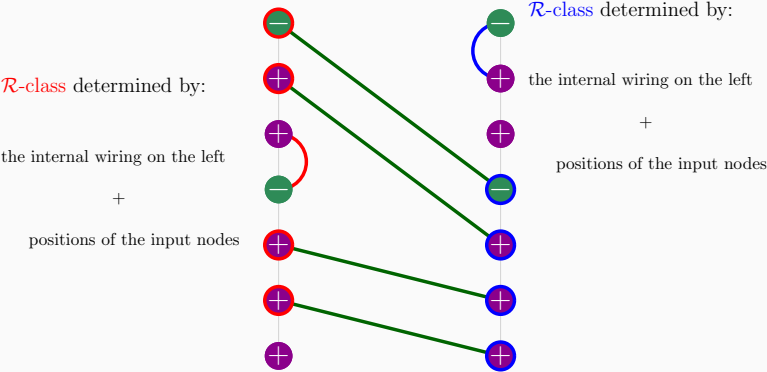the internal wiring on the left

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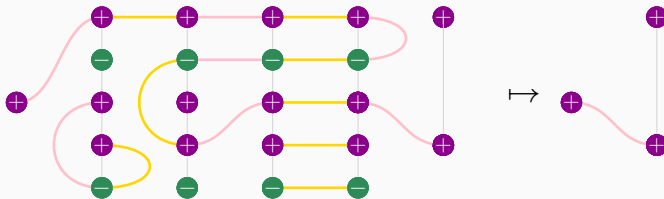- Planarity restriction is essential (consider)

Non-planar diagrams (with crossings): reminiscent of runs in 2DFAs!

- Transition functions $\delta : \Sigma \to \mathsf{Hom}(Q, Q)$ for some object Q $Q \approx$ set of directed states
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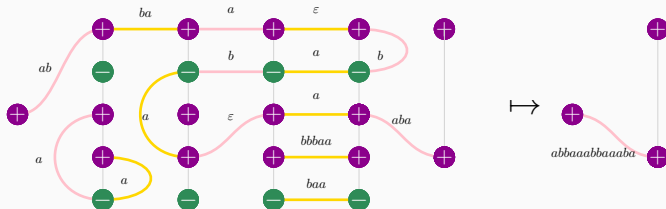- (links between GoI and planar 2DFAs already considered by (Hines 2003))

Theorem

Star-free languages are exactly those recognized by planar 2DFAs.

More generally: first-order transductions

Consider a richer category of diagrams where edges are labelled by output words

(labels of compositions given by concatenation)

Much like before, corresponding notion of (planar) 2DFTs.

Theorem

First-order transduction (FO regular functions) are those computed by reversible planar 2DFTs.

- 2DFTs with aperiodic transition monoid = FO regular functions [Carton&Dartois 2015]

  (hence reversible planar 2DFTs ⊆ FO-transductions)

- FO transduction ⊆ reversible planar 2DFTs: closure by composition and Krohn--Rhodes

  (see http://nguyentito.eu/2021-01-links.pdf)