Type 2 uncomputability,
(polynomial functors)
and automata (!)

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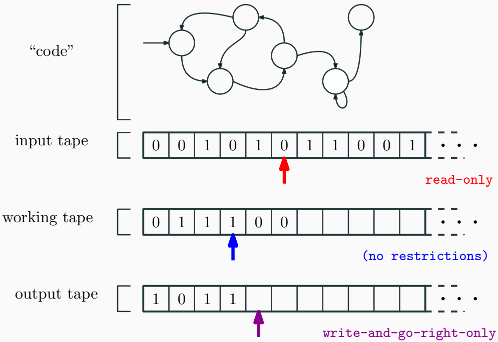
$\exists$ type-2 computability

The natural notion of computability for $2^{\mathbb{N}} \rightharpoonup 2^{\mathbb{N}}$



Formally, $f : 2^{\mathbb{N}} \rightharpoonup 2^{\mathbb{N}}$ computable

$$\iff \exists \sqsubseteq\text{-monotone computable } \widehat{f} : 2^* \to 2^* \text{ s.t. } f(p) = \lim_{n \to +\infty} \widehat{f}(p[n])$$

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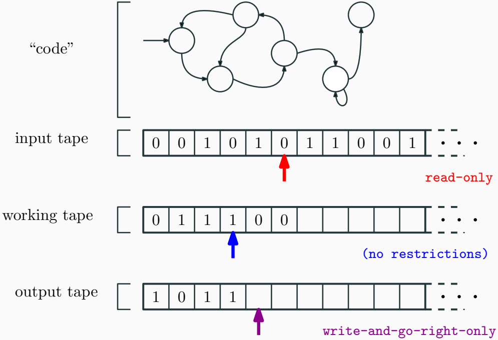

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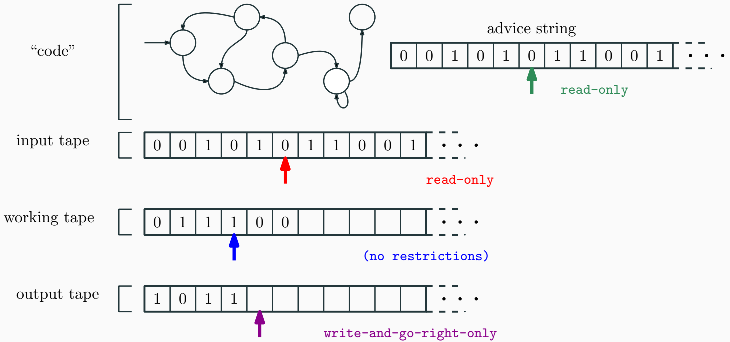
$\iff$ $\exists$ $\sqsubseteq$-monotone computable $\widehat{f} : 2^* \to 2^*$ s.t. $f(p) = \lim_{n \to +\infty} \widehat{f}(p[n])$

$\iff$ $\exists e \in \mathbb{N}.\ \forall n \in \mathbb{N}. \forall A \in \mathrm{dom}(f).\ \Phi_e^A(n) = f(A)(n)$

Continuity = type-2 computability relativized

Up to a type-1 oracle, all continuous functions are computable.



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$\iff$ $\exists e \in \mathbb{N}. \exists E \in 2^{\mathbb{N}}\ \forall n \in \mathbb{N}. \forall A \in \mathrm{dom}(f).\ \Phi_e^{E \oplus A}(n) = f(A)(n)$

Represented spaces

The category of represented spaces ReprSp

- Objects: (X, δ_X) where δ_X is a **partial** surjection $2^\omega \twoheadrightarrow X$
- Morphisms: maps $X \to X'$ with a type 2-computable witness

- Super nice: extensive, locally cartesian closed, (co)inductive types

Examples:

- Standard spaces like $\mathbb{N}$, $2^\mathbb{N}$, $\mathbb{N}^\mathbb{N}$

$$\delta_{2^\mathbb{N}} = \mathrm{id}_{2^\mathbb{N}} \qquad \delta_\mathbb{N}(0^n 1 p) = n \qquad \delta_{\mathbb{N}^\mathbb{N}}(0^{p(0)} 1 0^{p(1)} 1 \ldots) = p$$

- $\mathbb{R}$ via fast Cauchy sequences $\mathbb{Q}^\mathbb{N}$ (convergence rate 2^{-n}).
- More generally: all subspaces of (quasi/co)Polish spaces.

A setting for effective topology (1/2)

We can adapt basic definitions from synthetic topology.

The Sierpiński space $\mathbb{S}$

As a represented space, define $\delta_{\mathbb{S}} : 2^{\mathbb{N}} \to \{\bot, \top\}$ by

$$\delta_{\mathbb{S}}(x) = \bot \qquad \Longleftrightarrow \qquad x = 0^\omega$$

The spaces of open and closed subsets of X

We set $\mathcal{O}(X) = \mathcal{A}(X) = \mathbb{S}^X$, dual interpretations of $\in$

$$x \in U \in \mathcal{O}(X) \Longleftrightarrow U(x) = \top \qquad x \in F \in \mathcal{A}(X) \Longleftrightarrow F(x) = \bot$$

- Code for an open = eventually we notice if we are in
- Computably open = $\in$ is semi-computable
- $\bigcup : \mathcal{O}(\mathcal{O}(X)) \to \mathcal{O}(X)$, $\cup$, $\cap$, computable, ...

A setting for effective topology (2/2)

The space of compact subsets $\mathcal{K}(X)$

A subspace $\mathcal{K}(X) \subseteq \mathcal{O}(\mathcal{O}(X))$; concretely $\delta_{\mathcal{K}(X)}(p) = A \subseteq X$ iff

- we have $f \in \mathbb{S}^{\mathbb{S}^X}$ with $\delta_{\mathcal{O}(\mathcal{O}(X))}(p) = f$
- for all open $U \in \mathcal{O}(X)$, f detects if U covers A

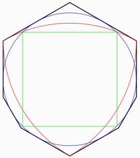
$$A \subseteq U \iff f(U) = \top$$

- X is computably compact iff $\forall_X : \mathcal{O}(X) \to \mathbb{S}$ is computable
- $[0,1]^\omega$, $2^{\mathbb{N}}$ computably compact...
- There exists a dual notion to compactness which is trivial classically, but relevant computably.

An application: Universal Lebesgue covering Problem

Theorem (Pauly & Ruggles '23)

The area of the smallest convex shape in $\mathbb{R}^2$ that cover all shapes of diameter 1 **is computable**.



Proof ideas:

- Show separately it is an upper/lower-semicomputable real
- $\inf U$ is upper-semi computable for a computable $U \in \mathcal{O}(\mathbb{R})$
- Use effective compactness and overtness over various things: compact subsets of $[0, 1]^2$, $\mathrm{GL}_2([-2, 2])$, ...

Non-computable, yet interesting tasks

- For given $x, y \in X$, do we have $x = y$ or not? $(=_X)$
- Given $p \in 2^{\mathbb{N}}$, does $p = 0^\omega$ or not? (LPO)
- Given $x, y \in \mathbb{R}$, do we have $x \leq y$ or $y \leq x$? (LLPO)
- Given a non-empty $A \in \mathcal{A}(X)$, find $x \in A$. (C_X)
- From $a, b \in \mathbb{R}$, find x such that $ax = b$. $(\mathsf{urbDiv}, \dots)$
- Given a colouring $[\mathbb{N}]^k \to n$, produce an homogeneous set. (RT^k_n)
- Find paths in various classes of ω-branching trees. $(\mathsf{WKL_0}, \dots)$
- $\dots$

Motivation for Weihrauch complexity

Compare the relative hardness of these tasks.

Weihrauch problems

Partial Weihrauch problems, formally

Families $(P_i)_{i \in \mathrm{dom}(P)}$ with $\mathrm{dom}(P) \subseteq 2^{\mathbb{N}}$ and $P_i \subseteq 2^{\mathbb{N}}$ for every i.

($\simeq$ internal families in regular projective represented spaces)

- $\mathrm{dom}(P)$ = set of **questions**
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- LPO: "Given $p \in 2^{\mathbb{N}}$, return 0 iff $p = 0^\omega$":

  $\mathsf{LPO}_{0^\omega} = \{0^\omega\}$ $\qquad$ $\mathsf{LPO}_p = \{10^\omega\}$ $\qquad$ otherwise

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- $\mathsf{C}_{\mathbb{N}}$, up to messing with representations for $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$:

$$\begin{array}{lcl} \mathrm{dom}(\mathsf{C}_{\mathbb{N}}) & = & \{p \in \mathbb{N}^{\mathbb{N}} \mid \exists n \in \mathbb{N}.\ n \notin \mathsf{range}(p)\} \\ (\mathsf{C}_{\mathbb{N}})_p & = & \{n \in \mathbb{N} \mid n \notin \mathsf{range}(p)\} \end{array}$$

Weihrauch reducibility

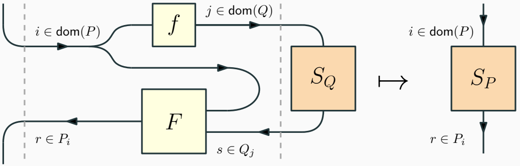
TL;DR: almost like Turing reducibility, but

- adapted to type 2 computability
- reductions must make **exactly** one oracle call
- (and Weihrauch problems are "non-deterministic")

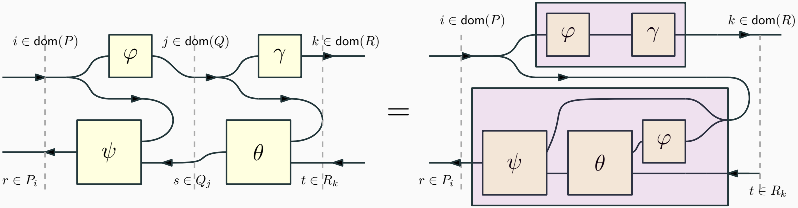
Official definition

$P \leq_{\mathrm{W}} Q$ if there are **computable**

$$f : \mathrm{dom}(P) \to \mathrm{dom}(Q) \quad \text{and} \quad F : \prod_{i \in \mathrm{dom}(P)} (Q_{f(i)} \to P_i)$$



Sanity checks



Reductions compose + Quotienting by $\equiv_{\mathrm{W}}$ $\rightsquigarrow$ **Weihrauch degrees**

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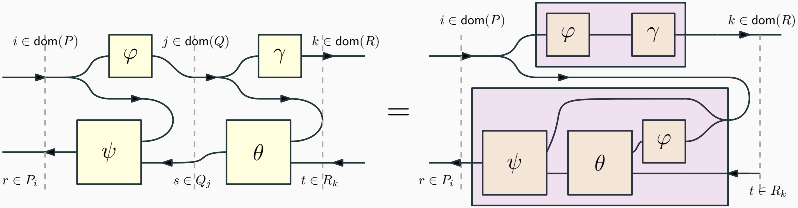

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Abstractly: reductions are container morphisms

Weihrauch problems = containers over represented subspaces of $2^{\mathbb{N}}$

Makes sense in over any category with pullbacks

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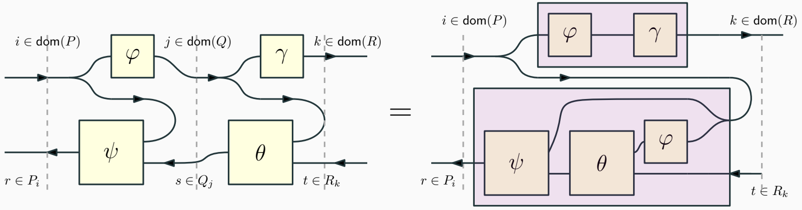

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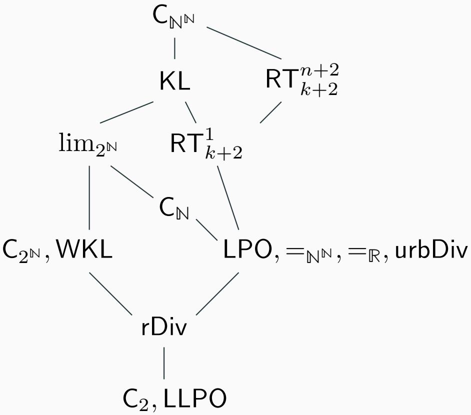
For LCCCs: "reductions" $\cong$ strong natural transformations between polynomial functors $X \longmapsto \sum_{i \in \mathrm{dom}(P)} X^{P_i}$

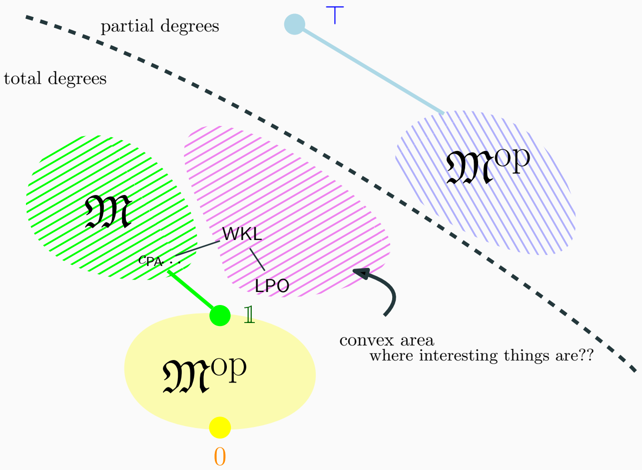
How some problems we've seen so far relate



One can certainly draw more ambitious zoos...

Zooming out

The (partial) Weihrauch lattice is big: $2^{2^{\aleph_0}}$.



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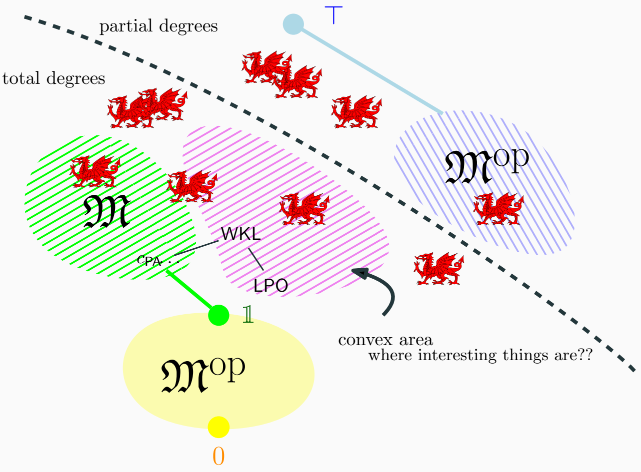

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Technical asides: encoding and intensionality

Legitimate question: why subsets of $\mathbb{N}^{\mathbb{N}}$?

Why not take internal families/containers in *represented spaces*?

- It would make sense and the category is nicer
- But computability theorists want reductions to be **intensional**, which yields more identifications than working extensional.

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Tradeoffs

- Containers over a **weakly** locally cartesian closed category
- "Wlog": space $\rightsquigarrow$ the domain of its representation

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Rough take-away: a lot of **algebra** from the theory of polynomial functors works, **up to annoying details**

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Rough take-away: a lot of **algebra** from the theory of polynomial functors works, **up to annoying details**

(or developping the theory of polynomial quasifunctors if you're volunteering)

Algebras of problems

Some algebra on problems

A lot of generic nice structure.

- Joins $+$: "solve one of the two problems"

$$\mathrm{dom}(P+Q) \cong \mathrm{dom}(P) + \mathrm{dom}(Q) \qquad \begin{array}{rcl} (P+Q)_{\mathsf{in}_1(i)} & = & P_i \\ (P+Q)_{\mathsf{in}_2(j)} & = & Q_j \end{array}$$

- Meets $\wedge$: "given inputs for both, solve one"

$$\mathrm{dom}(P \wedge Q) \cong \mathrm{dom}(P) \times \mathrm{dom}(Q) \quad (P \wedge Q)_{i,j} = P_i + Q_j$$

- Parallel product $\otimes$: "solve both problems"

$$\mathrm{dom}(P \otimes Q) \cong \mathrm{dom}(P) \times \mathrm{dom}(Q) \quad (P \otimes Q)_{i,j} = P_i \times Q_j$$

Exercise: what are the units? What are the corresponding operations on polynomials?

Some examples

- Most relationships with $+, \wedge$

  $\rightsquigarrow$ deduce from the theory of distributive lattices.
- $\mathsf{LPO} <_{\mathrm{W}} \mathsf{LPO} \otimes \mathsf{LPO} <_{\mathrm{W}} \ldots <_{\mathrm{W}} \mathsf{LPO}^n$
- $\mathsf{LPO}^n <_{\mathrm{W}} \mathsf{RT}^2_2$
- Some problems P are **finitely parallelizable**:

$$P \otimes P \leq_{\mathrm{W}} P \qquad \text{and} \qquad 1 \leq_{\mathrm{W}} P$$

  Examples: $\mathsf{WKL_0}$, $\lim_{2^{\mathbb{N}}}$, $\mathsf{C}_{\mathbb{N}}$, KL, $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

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Definition (?)

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$$P^* \quad = \quad \mu X.\, 1 + P \otimes X$$

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$$\mathrm{dom}(P^*) = \mathrm{dom}(P)^* \qquad\qquad P_w = \prod_{i \in |w|} P_{w_i}$$

One oracle call after the other

Sequential composition $P \star Q$

- Implicitly: ability to make an oracle call to Q then P
- Explicitly: given an instance i of Q and a function that takes a solution of i to an instance of P, compute all relevant solutions

$$\begin{aligned} \mathrm{dom}(P \star Q) &= \sum_{i \in \mathrm{dom}(Q)} \left(Q_i \xrightarrow{\leadsto} \mathrm{dom}(P)\right) \\ (P \star Q)_{i,f} &= \sum_{r \in Q_i} P_{f(r)} \end{aligned}$$

(using the weak exponential $\xrightarrow{\leadsto} \ldots$)

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\end{aligned}
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Extensional version, on polynomials

$$
\left(\sum_{i \in \mathrm{dom}(P)} X^{P_i}\right) \circ \left(\sum_{j \in \mathrm{dom}(Q)} X^{Q_j}\right) \cong \sum_{\substack{j \in \mathrm{dom}(Q) \\ f : Q_j \to \mathrm{dom}(P)}} X^{\sum_r P_{f(r)}}
$$

Some examples

- Obviously, $P \otimes Q \leq_{\mathrm{W}} P \star Q$.
- $\mathsf{LPO} \otimes \mathsf{LPO} <_{\mathrm{W}} \mathsf{LPO} \star \mathsf{LPO} <_{\mathrm{W}} \mathsf{LPO} \otimes \mathsf{LPO} \otimes \mathsf{LPO}$
- Some problems P are closed under sequential composition

$$P \star P \leq P \qquad\qquad 1 \leq P$$

Examples: $\mathsf{C}_{\mathbb{N}}, \mathsf{WKL_0}, \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

Iterated composition $P^\diamond$

- Explicitly: computed as the least fixpoint of $X \mapsto \mathbb{1} + X \star P$
- Implicitly: ability to make a finite but not fixed in advance number of oracle calls to P

Automata link 1:
Fixpoints of problems

Motivation

We've seen some useful **least fixpoints**.

Other useful fixpoints

Infinite parallelization $\widehat{P}$: fixpoint of $X \mapsto P \otimes X$

$$\mathrm{dom}(\widehat{P}) \quad = \quad \mathrm{dom}(P)^{\mathbb{N}} \qquad\qquad \widehat{P}_s \quad = \quad \prod_{n \in \mathbb{N}} P_{s_n}$$

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Construction as an inverse limit left to the listener

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Exercise: these are **not** greatest fixpoints!

E.g. $\nu X.\, \mathsf{LPO} \otimes X \quad \equiv_{\mathrm{W}} \quad \nu X.\, X \star \mathsf{LPO} \quad \equiv_{\mathrm{W}}$

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Fixpoint of operators

Existence of useful fixpoints (P. & Price '26)

If F is a nice enough endofunctor over containers, over a nice enough category (whatever that means), the following exists:

- an initial algebra μF for F
- a terminal coalgebra νF for F
- a somewhat canonical (co)algebra ζF sitting in-between

- Idea: construct a fixpoint on domains, and then on the answers.
- Application: $\widehat{P} \equiv_{\mathrm{W}} \zeta X.\ X \otimes P$ and P^{ω} morally like $\zeta X.\ X \star P$.

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Meta-question (I don't know the answer)

Is this relevant for other applications of containers?

Towards an automata-theoretic problem

We were drawn to studying (total parts) of problems generated by the following closed ζ-expressions:

$$P, Q ::= X \mid \gamma X.P \mid P \mathbin{\Box} Q \qquad \Box \in \{+, \otimes, \wedge\}, \gamma \in \{\mu, \nu, \zeta\}$$

They're all generated by regular infinite **game-tree** languages.

Game-tree problems

Given an infinite regular tree language

$$L \subseteq \mathsf{Tree}(\{\exists, \forall\} \times \{0, \ldots, k\}) \qquad \text{(a tree = an arena for a parity game)}$$

define the problem $[\![L]\!]$ by $\mathrm{dom}([\![L]\!]) = L$ and

$$[\![L]\!]_G \quad = \quad \text{winning strategies for Eve in } G$$

Most problems we've seen today are equivalent to game-tree problems.

A purely automata-theoretic question?

Conjecture

From automata recognizing L, L', we can decide "$[\![L]\!] \leq_{\mathrm{W}} [\![L']\!]$?".

- Very easy exercise: with $[\![L]\!] \equiv_{\mathrm{W}} \mathbb{1}$
- Maybe exercise (???): with $[\![L']\!] \equiv_{\mathrm{W}} \mathbb{1}$
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- ($\exists$ a variant of the conjecture in trees with coloured holes in the paper)

Automata link 2: Variants of Kleene algebras

Equational theory of the Weihrauch lattice

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, +, \wedge)$

  (via the Medvedev degrees)
- Thus, $(\mathfrak{W}, +, \wedge) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
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Meta-question

For a given signature, is there anything true in the Weihrauch degree that is not true for **all** (suitable) categories of containers?

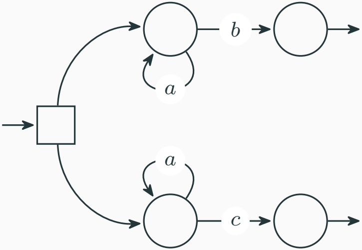
Terms with composition and automata

Starting observation

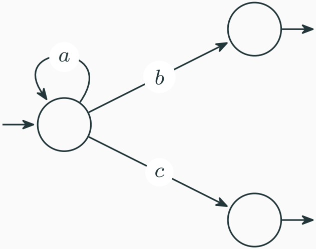
Terms over $0, \mathbb{1}, +, \star, (-)^\diamond, \top$ = almost regular expressions.

(alphabet = the set of variables)

- Terms can be mapped to NFAs in a meaningful way
- Adding $\wedge$ = allowing alternating automata



$b \star a^\diamond \;\wedge\; c \star a^\diamond$



$(b + c) \star a^\diamond$

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\eth(\mathcal{A}, \mathcal{B})$ that captures a variant of the notion of **simulation** such that

Theorem

$(\mathfrak{W}, \mathbb{1}, +, \top, \wedge, \star, (-)^\diamond) \models t \leq u$ iff Duplicator wins in $\eth(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\eth(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
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(using $\mathcal{B}$ exactly once)

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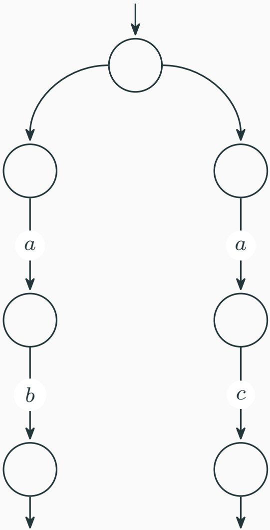
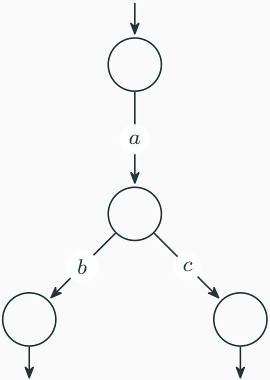
Corollary

"$(\mathfrak{W}, \mathbb{1}, 0, \top, +, \wedge, \star, (-)^{\diamond}) \models t \leq u$?" is decidable.

- Conjecture: this is PSPACE-complete.

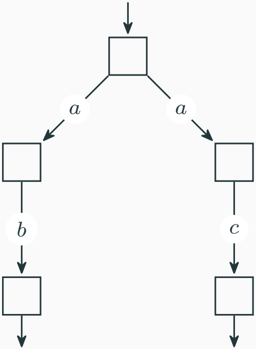
A simple example of simulation and non-simulation

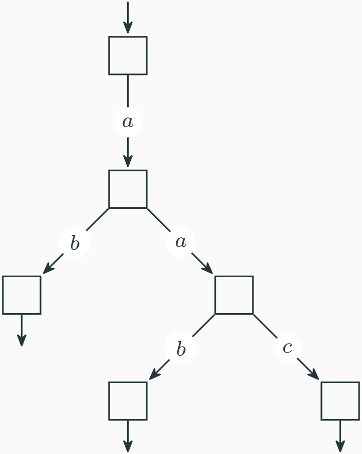




$$(b \star a) + (c \star a) \quad < \quad (b + c) \star a$$

A simulation requiring several concurrent attempts





$$(b \star a) \wedge (c \star a) \qquad < \qquad (((c \wedge b) \star a \;\wedge\; b) \star a$$

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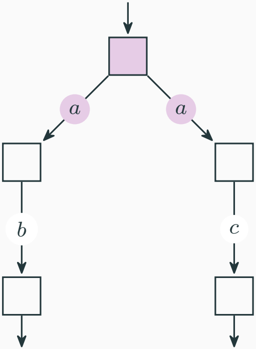

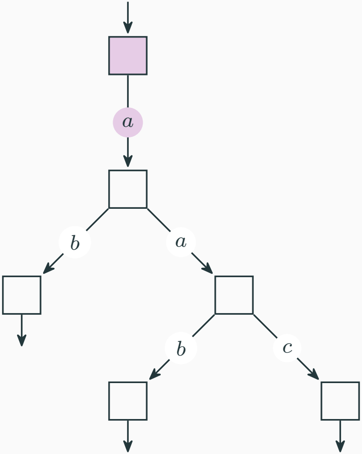

$(b \star a) \wedge (c \star a) \qquad < \qquad (((c \wedge b) \star a \ \wedge\ b) \star a$

Let's play ?!

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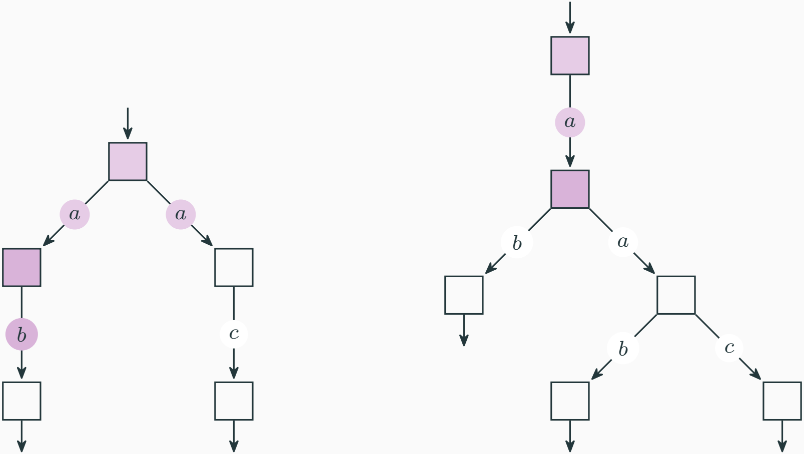

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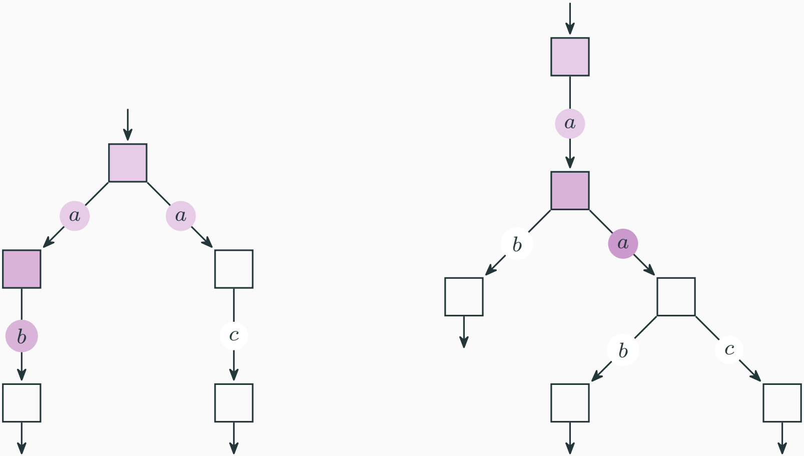

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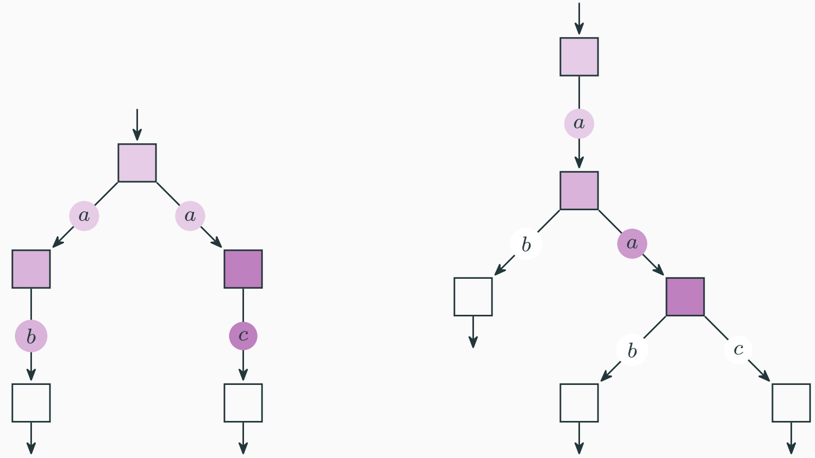

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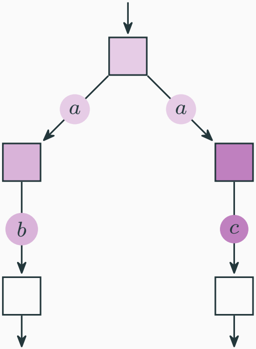

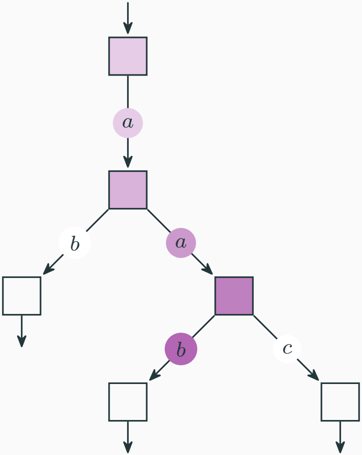

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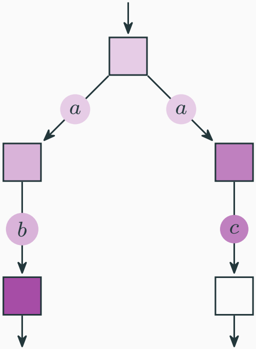

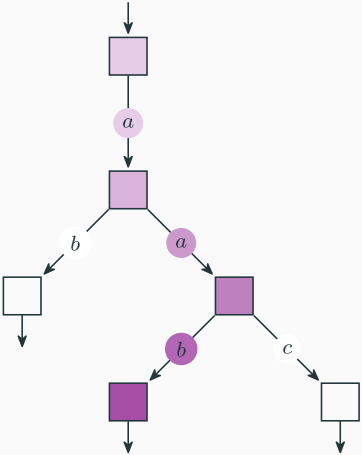

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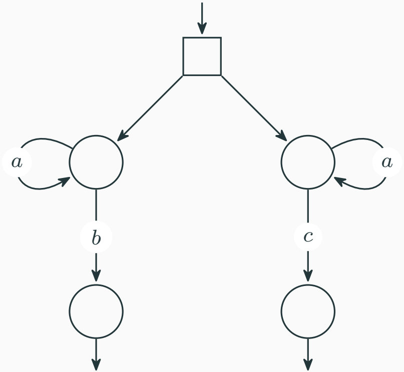
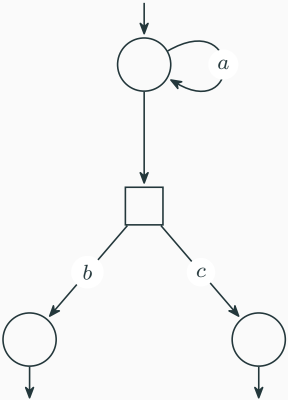
Another simulation requiring several concurrent attempts





$$(b \star a^\diamond) \wedge (c \star a^\diamond) \qquad \equiv \qquad (b \wedge c) \star a^\diamond$$

Where do the concurrent attempt come from

- $\mathrm{dom}(a \wedge b) = \mathrm{dom}(a) \times \mathrm{dom}(b)$,
- Solving either question is progress along a possible run
- Conceptually captured in the following **strength** axiom:

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For category theorists

Corresponds to the canonical strength in polynomial functors.

$$P(X) \times Y \quad \longleftarrow \quad P(X \times Y)$$

Induction principles for $(-)^{\diamond}$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \star x \leq x \qquad \Rightarrow \qquad x^{\diamond} \leq x$$

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(exercise: $\mathbb{1} \leq a \wedge b$ implies $a^\diamond \wedge b^\diamond \leq (a \wedge b)^\diamond$)

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Theorem

The above axioms are valid in the partial Weihrauch degrees.

Completeness

Candidate axiomatization of inequations

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(x\star y)\wedge z & \leq & (x\wedge z)\star y & \text{strength}\\
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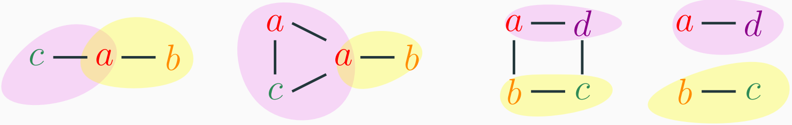
Proof idea: $\exists$ positional simulation strategies, induction on the syntax

How this started: the theory of $\otimes, \wedge$

A notion of combinatorial reduction between graphs

A reduction from (V_0, E_0, c_0) to (V_1, E_1, c_1) is a colour-preserving function $h : V_1 \to V_0$ such that the image of any maximal clique in (V_1, E_1) under h contains a maximal clique in (V_0, E_0).



$a \otimes (c \wedge b) \quad < \quad a \otimes ((a \otimes c) \wedge b) \qquad (a \wedge c) \otimes (b \wedge d) \quad < \quad (a \otimes d) \wedge (b \otimes c)$

Combinatorial characterization (Neumann, Pauly, P.)

$(\mathfrak{W}, \wedge, \otimes) \models t \leq u$ iff there is a reduction from G_t to G_u.
As a result, deciding $(\mathfrak{W}, \wedge, \otimes) \models t \leq u$ is Σ_2^p-complete.

- Axiomatizing: harder!
- $+$, $\star$ and $\otimes$: opens the gates of hell (**concurrency theory**)

Concluding thoughts on this stuff

I feel ignorant (help?)

- Is this relevant/known to non-computability theorists?
- Do people care about the algebra of strong monotone operators on complete Heyting algebras?
- Literature on simulation between alternating automata?

Avenues for further work

- Horn theory?
- Fun extensions to the signature:
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