Some proof-theoretical approaches to Monadic Second-Order logic

PhD defense

Cécilia Pradic

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Uniwersytet Warszawski

Goal : check safety of engineered systems

- "The green and red lights are not on at the same time"
- "Orange is flashed before red"
- ...

Verification of engineered systems/programs

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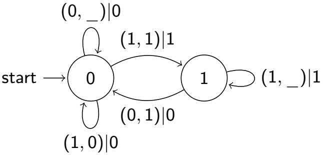
Some more complicated devices:

Typical system:

$\approx$



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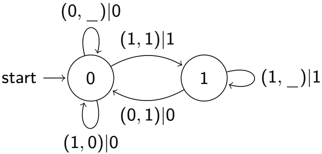

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Logic for verification

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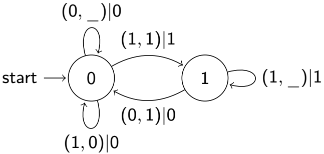

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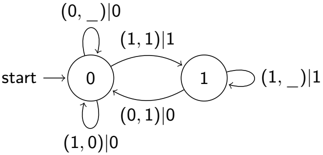

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Answer whether yes or no a system satisfies φ?

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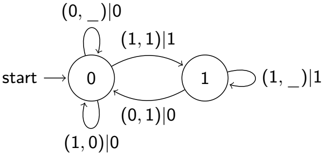

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Generate a system satisfying φ from scratch.

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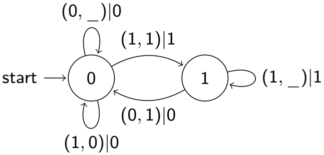

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Decide logic?

Logic and decidability, algorithmically

Hilbert's dream

The more extreme version of Hilbert's program (1920s):

- Reduce mathematics to formalized arithmetics.
- A mechanical method to decide the (mathematical) **truth**.

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Decidable subcases

- Logics over fixed finite domains.
- Monadic Second Order (MSO) logic over infinite words.

Proof theory and constructivity

Proof theory at a very high level

Formalize mathematically what is a correct proof.

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$\pi + e$ is transcendental or $e \cdot \pi$ is transcendental (or both are).

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A *constructive proof* would be more informative.

$$\text{proofs} \longrightarrow \text{computable witnesses}$$

Monadic Second-Order (MSO) logic and constructiveness

Monadic Second-Order logic (MSO)

- A fragment of Second-Order logic.
- Algorithmically decidable over

  $\mathbb{N}, \mathbb{Q}$, the infinite binary tree $\{0,1\}^*$, ...
- Subsumes many verification logics. LTL, CTL, ...

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Decidable $\neq$ constructive

Soundness of decision procedures $\Longleftarrow$ non-constructive theorems.

- Over $\mathbb{N}$: infinite Ramsey theorem, weak König's Lemma.
- Over $\{0,1\}^*$: determinacy of infinite parity games.

How (non)-constructive is MSO?

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What axiomatic strength characterizes a given MSO *theory?*

- With H. Michalewski, L. Kołodziejczyk and M. Skrzypczak in Warsaw.

When can we extract computational content from MSO *proofs?*

- With C. Riba in Lyon.

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$\rightsquigarrow$ Metatheoretical analysis of Büchi's decidability theorem.

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- With C. Riba in Lyon.

$\rightsquigarrow$ Refinement of MSO($\mathbb{N}$) with witness extraction.

Outline

Monadic Second-Order logic

Part I: Reverse Mathematics

Part II: proof systems for Church's synthesis

Conclusion

MSO over infinite words

Syntax of MSO($\mathbb{N}$)

$$\varphi, \psi ::= n \in X \mid n < k \mid \exists n\, \varphi \mid \exists X\, \varphi \mid \neg\varphi \mid \varphi \wedge \psi$$

- Can be regarded as a subsystem of Second-Order Arithmetic
- Standard model: $n \in \mathbb{N}$, $X \in \mathcal{P}(\mathbb{N})$
- Only *unary* predicates. no pairing, no addition

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Typical MSO($\mathbb{N}$)-definable properties

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Corresponds exactly to sets recognizable by automata over infinite words.

- Infinite words: regard sets as sequences of bits through $\mathcal{P}(\mathbb{N}) \simeq 2^\omega$
- $\varphi(X_1, \ldots X_k)$: formula over Σ^ω for $\Sigma = 2^k$

Non-deterministic Büchi automata (NBA)

Definition

A non-deterministic Büchi automaton (NBA) $\mathcal{A} : \Sigma$ is a tuple (Q, q_0, δ, F)

- Q is a finite set of states, $q_0 \in Q$
- transition function $\delta : \Sigma \times Q \to \mathcal{P}(Q)$
- $F \subseteq Q$ *accepting states*

Recognizes languages of infinite words $\mathcal{L}(\mathcal{A}) \subseteq \Sigma^\omega$:

$w \in \mathcal{L}(\mathcal{A})$ iff there is a run over $w \in \Sigma^\omega$ hitting F infinitely often

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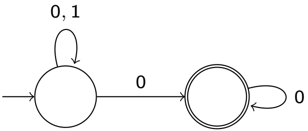
Example:



$\mathcal{L}(\mathcal{A}) =$ streams with finitely many 1.

MSO/automata correspondance

$$\begin{tikzcd}
\text{MSO formulas over } \Sigma \arrow[rr, bend left, "\varphi \mapsto \mathcal{A}_\varphi"] \arrow[dr, "\varphi \mapsto \mathcal{L}(\varphi)"'] & & \text{automata over } \Sigma \arrow[dl, "\mathcal{A} \mapsto \mathcal{L}(\mathcal{A})"] \\
& \mathcal{P}(\Sigma^\omega) &
\end{tikzcd}$$

Decidability [Büchi (1962)]

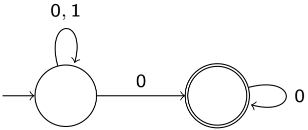
MSO over infinite words is decidable.

- Proof idea: automata theoretic-construction for each logical connective.
- Hard case for infinite words: negation $\neg$.

corresponds to *complementation*

Complementation, determinization and constructivity

For finite word automata: easy complementation for *deterministic* automata.



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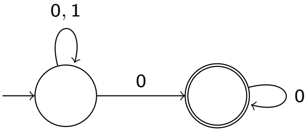

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Theorem [McNaughton (1968)]

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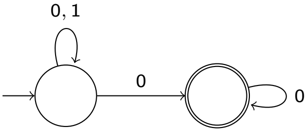

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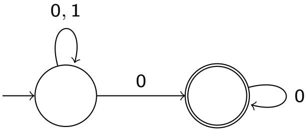

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Quantify how non-constructive they are?

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- Reverse Mathematics
- Büchi's theorem
- Beyond infinite words

Part II: proof systems for Church's synthesis

Conclusion

Reverse Mathematics

- A framework to analyze axiomatic strength.
- Vast program. [Friedman, Simpson, Steele 70s]

Methodology

- Consider a theorem T formulated in second-order arithmetic.
- Work in the weak theory RCA_0.
- Target some natural axiom A such that $\mathrm{RCA}_0 \nvdash A$.
- Show that $\mathrm{RCA}_0 \vdash A \Leftrightarrow T$.

Essentially independence proofs. . .

- Similar in spirit to statements like

  "Tychonoff's theorem is equivalent to the axiom of choice."

The big five

Π^1_1 Comprehension	Π^1_1–CA_0	$\Longleftrightarrow$	Lusin's separation theorem
	$\Downarrow$		
Transfinite Recursion	ATR_0	$\Longleftrightarrow$	Determinacy of open games
	$\Downarrow$		
Σ^0_1 Comprehension	ACA_0	$\Longleftrightarrow$	König's Lemma
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Weak König's Lemma	WKL_0	$\Longleftrightarrow$	Brouwer's fixed point theorem
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Outliers: infinite Ramsey for pairs, determinacy statements.

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$\rightsquigarrow$ Where does Büchi's theorem sit in this hierarchy?

$$\begin{tikzcd}
\text{Weak König's lemma} \arrow[ddd, Rightarrow] & & & \text{Infinite Ramsey theorem} \arrow[d, Rightarrow] \\
& \Sigma^0_2\text{-induction} \arrow[rr, Rightarrow] & & \text{Additive Ramsey} \arrow[d, Rightarrow] \\
& \mathsf{MSO}(\omega) \arrow[u, Rightarrow] \arrow[dl, Rightarrow] & & \text{Compl. of NBA} \arrow[ll, Rightarrow] \arrow[d, Rightarrow] \\
\text{Bounded weak König's lemma} & & & \text{Determinization of NBA}
\end{tikzcd}$$

The Logical Strength of Büchi's Decidability Theorem

[Kołodziejczyk, Michalewski, P., Skrzypczak, 2016]

Beyond infinite words

Theorem [Kołodziejczyk, Michalewski (2015)]

Decidability of MSO over the infinite binary tree is not provable in Π^1_2-CA_0.

- Rabin's theorem requires much higher axiomatic strength.
  - Roughly on par with determinacy of infinite parity games.

$BC(\mathbf{\Sigma^0_2})$ games

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- Intermediate cases?

MSO over the rationals (MSO($\mathbb{Q}$))

- Decidable via a reduction to the infinite tree.
- Cover all countable linear orders.
- Direct algebraic decidability proofs.

[Shelah (1975)], [Carton, Colcombet, Puppis (2013)]

Strength of additive Ramsey over $\mathbb{Q}$ and MSO($\mathbb{Q}$)

Theorem [Kołodziejczyk, Michalewski, P., Skrzypczak]

Over RCA_0, the following are equivalent:

- the shuffle principle [Carton, Colcombet, Puppis (2013)]
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The classical theory MSO($\mathbb{Q}$) has a sentence equivalent to Π^1_1-CA_0.

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Conjecture

Over RCA_0, the following are equivalent:

- The axiom of finite Π^1_1-recursion.
- Determinacy of infinite weak parity games. $BC(\mathbf{\Sigma^0_1})$ games
- Soundness of the decision algorithm for MSO($\mathbb{Q}$).

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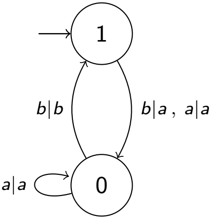
Part II: proof systems for Church's synthesis
- Church's synthesis and witness extraction
- Constructive proof systems
- Categorical/syntactic approach

Conclusion



Causal/synchronous stream functions $f : \Sigma^\omega \to \Gamma^\omega$

- Interpret $n \in \mathbb{N}$ as **time steps**.
- Lifted from functions $\hat{f} : \Sigma^+ \to \Gamma$ as

$$\begin{array}{llll} \hat{f} : & \Sigma^\omega & \to & \Gamma^\omega \\ & s & \mapsto & n \mapsto f(s(0)\ldots s(n)) \end{array}$$

  i.e., the output does not depend on the future.
- Focus on *finite-state* causal functions.

(Correspond to *Mealy machines*)

- All f.s. causal functions are recursive.
- All causal functions are continuous.
- Some recursive functions are not causal. $w \longmapsto n \mapsto w_{n+1}$

Church's synthesis (2/2): the Büchi-Landweber theorem

Church's synthesis problem

Given a formula $\varphi(X, Y)$, find a f. s. causal $f : \Sigma^\omega \to \Gamma^\omega$ such that

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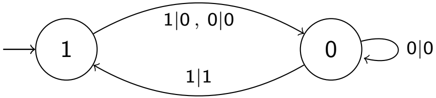
Example (inspired from [Thomas (2008)]):

- $\varphi(X, Y) \quad \equiv \quad (X \text{ infinite} \Rightarrow Y \text{ infinite}) \quad \text{and} \quad \forall i \, (i \in Y \Rightarrow i + 1 \notin Y)$



Theorem [Büchi-Landweber (1969)]

Algorithmic solution for $\varphi(X, Y)$ in MSO.

- Algorithmically costly...

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Classical theorems in MSO

- Excluded middle (subtle point $\{0,1\}^\omega$ vs $\mathcal{P}(\mathbb{N})$)
- The infinite pigeonhole principle
- Instances of additive Ramsey

$\rightsquigarrow$ No algorithmic witnesses for $\forall\exists$ theorems.

Extraction from proofs

Goal: a refinement of MSO($\mathbb{N}$) with extraction for **causal** functions.

- Toward semi-automatic approach to synthesis.
- Approach inspired by realizability. [Kleene (1945), ...]

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Analogous example: extraction for intuitionistic arithmetic (HA)

If $HA \vdash \forall x \exists y \varphi(x, y)$, there is an algorithm computing

$$f : \mathbb{N} \to \mathbb{N} \text{ recursive} \qquad \text{such that} \qquad \forall x \quad \varphi(x, f(x))$$

- A subset of classical arithmetic (PA).
- As expressive as classical arithmetic. ($\varphi \mapsto \varphi^{\neg\neg}$)
- Can be refined to System T functions. [Gödel (1930s)]

Analogy

Classical system	MSO($\mathbb{N}$)	PA
Realizers	Causal functions	System T
Intuitionistic system	???	HA

Synchronous MSO (SMSO) [P., Riba (2017)]

Intuitionistic version of MSO

$$\varphi, \psi \quad ::= \quad \alpha \;\mid\; \varphi \wedge \psi \;\mid\; \exists X\, \varphi \;\mid\; \neg\varphi$$

Quantification over individuals encoded as usual

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Glivenko's theorem for SMSO

MSO $\vdash \varphi$ if and only if SMSO $\vdash \neg\neg\varphi$

- Negation erases computational contents.

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Extraction of f.s. causal functions

SMSO $\vdash \exists y\ \neg\neg\varphi(x, y)$ iff there is a f.s. causal f s.t. MSO $\vdash \forall x\ \varphi(x, f(x))$

- Proofs $\varphi \vdash \psi$ interpreted as simulations between ND automata.

Synchronous MSO (SMSO) [P., Riba (2017)]

Intuitionistic version of MSO

$$\varphi, \psi \quad ::= \quad \alpha \ \mid \ \varphi \wedge \psi \ \mid \ \exists X\ \varphi \ \mid \ \neg\varphi$$

Quantification over individuals encoded as usual

Glivenko's theorem for SMSO

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- Negation erases computational contents.

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Polarity restriction

A linear refinement LMSO [P., Riba (2018)]

- Polarized system with dualities.
- Requires the introduction of **linear** connectives.

Linear MSO (LMSO)

$$\varphi, \psi \quad ::= \quad \alpha \ | \ \varphi \otimes \psi \ | \ \varphi \parr \psi \ | \ \varphi \multimap \psi \ | \ \forall X \varphi \ | \ \exists X \varphi \ | \ !\varphi^- \ | \ ?\varphi^+ \ | \ \ldots$$

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Alternating $(\forall, \exists, \otimes, \parr, \multimap)$

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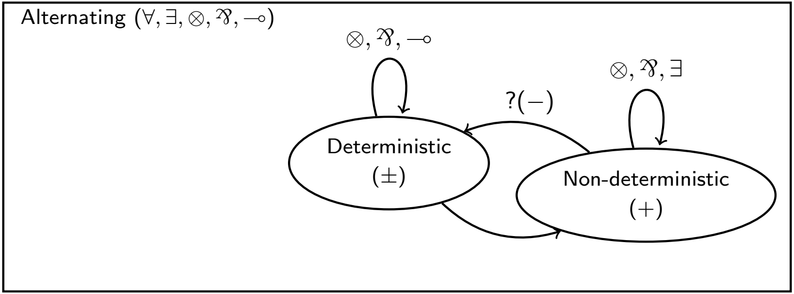

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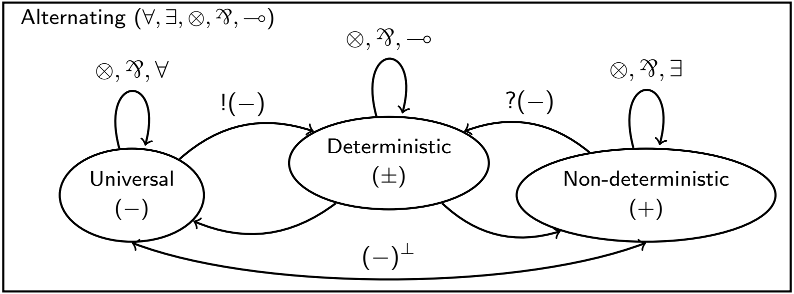

SMSO $\approx$ restriction to positives

Expressivity and proof extraction for LMSO

Conservativity

$$\begin{array}{ccc} \mathsf{LMSO} & \to & \mathsf{MSO} \\ \varphi & \mapsto & \lceil \varphi \rceil \end{array}$$

If LMSO $\vdash \varphi$, then MSO $\vdash \lceil \varphi \rceil$.

Expressivity

$$\begin{array}{ccc} \mathsf{MSO} & \to & \mathsf{LMSO} \\ \varphi & \mapsto & \varphi^L \end{array}$$

If MSO $\vdash \varphi$, then LMSO $\vdash \varphi^L$.

$$\begin{tikzcd}
\text{LMSO} \arrow[rr, bend left, "\varphi \mapsto \mathcal{A}_\varphi"] \arrow[dr, "\varphi \mapsto [\![\varphi]\!]"'] & & \text{Alternating automata} \arrow[dl, "\text{Acceptance game}"] \\
& \text{Simulation games} &
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- LMSO includes *Full Intuitionistic Multiplicative Linear Logic*.
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$$\forall x \in \Sigma^\omega\ \exists y \in \Gamma^\omega\ \varphi(x,y) \quad \multimap \quad \exists f \in (\Sigma \to \Gamma)^\omega\ \forall x \in \Sigma^\omega\ \varphi(x, f(x))$$

$f(x)$ for pointwise application

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For every φ, there is a realizer $\quad (\varphi \multimap \bot) \multimap \bot \quad \multimap \quad \varphi$

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Double linear-negation elimination

For every φ, there is a realizer $\quad (\varphi \multimap \bot) \multimap \bot \quad \multimap \quad \varphi$

but no canonical iso in general!

- Also holds in DC if the base satisfies choice.

Why automata?

The above logic can be defined without reference to automata.

- ω-word automata guarantee decidability properties. . .
- But they are not needed to extract realizers.

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The above logic can be defined without reference to automata.

- ω-word automata guarantee decidability properties...
- But they are not needed to extract realizers.

$\rightsquigarrow$ A purely logical reformulation of LMSO using categorical semantics.

Goals

- Purely syntactic transformations.
- Understand links with typed realizability and Dialectica.

Finite-state causal functions as terms

Define the category $\mathbb{M}$ of causal functions

- Objects: sets of streams Σ^ω for Σ finite
- Morphisms: finite-state causal functions
- Cartesian products $\Sigma^\omega \times \Gamma^\omega \simeq (\Sigma \times \Gamma)^\omega$, but **not** cartesian-closed

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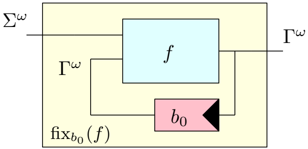
Inductive presentation

$$\frac{f : \Sigma \to \Gamma}{f^\omega : \Sigma^\omega \to \Gamma^\omega} \qquad \frac{f : \Sigma^\omega \times \Gamma^\omega \to \Gamma^\omega \quad b_0 \in \Gamma}{\mathrm{fix}_{b_0}(f) : \Sigma^\omega \to \Gamma^\omega}$$

+ closure under composition



$\approx$ guarded recursion $\mathrm{fix} : A^{\blacktriangleright A} \to A$

topos of trees

MSO($\mathbb{N}$) as an equational logic over $\mathbb{M}$

FOM (First-Order Mealy)

$$\varphi, \psi \quad ::= \quad \mathrm{t} =_{\Sigma^\omega} \mathrm{u} \;\mid\; \varphi \wedge \psi \;\mid\; \neg\varphi \;\mid\; \exists x \in \Sigma^\omega\ \varphi$$

- Typed variables stand for streams, terms for every f.s. causal functions.

Proposition

FOM and MSO($\mathbb{N}$) are interpretable in one another.

- Justifies focusing on FOM.

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Tarskian semantics (categorical logic)

- Regard $\mathbb{M}$ as a multi-sorted Lawvere theory.

$\rightsquigarrow$ Tarskian semantics $\approx$ indexed category, from global section functor $\mathbf{\Gamma}$

$$\begin{aligned} \mathbf{\Gamma} : \quad \Sigma^\omega &\longmapsto \mathrm{Hom}_{\mathbb{M}}\left(1^\omega, \Sigma^\omega\right) \\ \Sigma^\omega &\longmapsto \left(\mathcal{P}\left(\mathbf{\Gamma}\left(\Sigma^\omega\right)\right), \subseteq\right) \end{aligned}$$

SMSO and the simple fibration

Simple slice $\mathcal{C}//X$ = full subcategory of $\mathcal{C}/X$ with objects

$$X \times Y \xrightarrow{\pi} X$$

$\rightsquigarrow$ the simple fibration $\mathsf{s}(\mathcal{C}) \to \mathcal{C}$

The construction $\mathfrak{Sum}$

$$\begin{tikzcd}
\mathfrak{Sum}(\mathcal{E}) \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\mathfrak{Sum}(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathsf{s}(\mathcal{C}) \arrow[r, "\times"] \arrow[d] & \mathcal{C} \\
\mathcal{C} &
\end{tikzcd}$$

- $\mathfrak{Sum}(p)$-predicate: $(U, \varphi(a, u))$
  U object of $\mathcal{C}$, φ over $A \times U$ (in p)
  $\approx \exists u : U\ \varphi(a, u)$
- Freely adds existential quantifications
  (simple sums)
- Reminiscent of typed realizability
  realizers in $\mathcal{C}$

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Reconstructing SMSO

Simulations of non-determinstic automata $\approx$ $\mathfrak{Sum}$ applied to FOM

Fibered Dialectica [Hyland (2001)]

$\mathfrak{Dial} \cong \mathfrak{Sum} \circ \mathfrak{Prod}$ $\mathfrak{Prod}(p) \cong \mathfrak{Sum}(p^{\mathbf{op}})^{\mathbf{op}}$ [Hofstra (2011)]

- $\mathfrak{Dial}(p)$-predicate over $A \approx (U, X, \varphi(a, u, x))$

  think $\exists u \; \forall x \; \varphi(a, u, x)$

- interprets full intuitionistic MLL+FO

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Linking LMSO with Dialectica

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- Polarity restrictions $\approx$ model of LMSO (restricted exponentials)

Outline

Monadic Second-Order logic

Part I: Reverse Mathematics

Part II: proof systems for Church's synthesis

Conclusion

Summary

Axiomatic strength of two classical MSO theories.

- In the context of Reverse Mathematics.
- Strong link between Σ^0_2-induction and MSO($\mathbb{N}$).
- Preliminary results on MSO($\mathbb{Q}$).

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Related work

- Characterizations of the topological complexity of MSO-definable sets.
- Extension to the Reverse-mathematical analysis to intuitionistic logic.
  [Lichter and Smolka (2018)]
- Conservativity results for cyclic arithmetic.
  [Simpson (2017), Das (2019)]

Summary

- Realizability models based on simulations between automata
- Abstract reformulation — link with Dialectica and typed realizability
- Complete extension of LMSO — omitted from the talk [P., Riba (2019)]

Part II: Curry-Howard for MSO($\mathbb{N}$)

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Related work

- Fibrations of tree automata — [Riba (2015)]
- Good-for-games automata — [Henziger, Piterman (2006), Kuperberg Skrzypczak (2015)]

Final word

Some further questions

- Realizability for *continuous* functions $\Sigma^\omega \to \Gamma^\omega$?
- Extensions of $\mathfrak{Dial}^{\blacktriangleright}$ for fibrations over the topos of trees?

  $\mathfrak{Fam}(\mathfrak{Fam}(p^{\mathbf{op}})^{\mathbf{op}})$ instead of $\mathfrak{Dial}(p)$
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Thanks for your attention! Questions?

Induction and comprehension

RCA_0 is defined by restricting *induction* and *comprehension*

Comprehension axiom

For every formula $\phi(n)$ (with $X \notin FV(\phi)$)

$$\exists X \; \forall n \in \mathbb{N} \quad (\phi(n) \Leftrightarrow n \in X)$$

- RCA_0: restricted to Δ^0_1 formulas

recursive comprehension

Induction axiom

To prove that $\forall n \in \mathbb{N} \phi(n)$ it suffices to show

- $\phi(0)$ holds
- for every $n \in \mathbb{N}$, $\phi(n)$ implies $\phi(n+1)$

- RCA_0: restricted to Σ^0_1 formulas.

$\exists n \; \delta(n)$ with $\delta \in \Delta^0_1$

- Equivalent to minimization principles and comprehension for finite sets.

Additive Ramsey over ω

For any linear order $(P, <)$ write $[P]^2$ for $\{(i, j) \in P^2 \mid i < j\}$ and fix a finite monoid $(M, \cdot, e)$.
Call $f : [P]^2 \to M$ *additive* when $f(i, j) \cdot f(j, k) \ = \ f(i, k)$ for all $i < j < k$

Additive Ramsey

For any additive $f : [P]^2 \to M$, there is an unbounded monochromatic $X \subseteq P$ (s.t. $|f([X]^2)| = 1$).

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For any additive $f : [P]^2 \to M$, there is an unbounded monochromatic $X \subseteq P$ (s.t. $|f([X]^2)| = 1$).

Theorem

Over RCA_0, additive Ramsey over ω is equivalent to Σ^0_2-induction.

Direct proof: "as usual" for additive Ramsey (factored through an ordered variant in the paper)

Π^0_2-induction from additive Ramsey

Consider equivalently comprehension for sets bounded by n for $\exists^\infty k \;\; \delta(x,k)$.
Define the coloring $f : [\omega]^2 \to 2^n$ as $f(i,j)_x = \max_{i \le l < j} \delta(x,l)$.

Apply additive Ramsey and consider the color X of the monochromatic set; we have

$$x \in X \qquad \Leftrightarrow \qquad \exists^\infty \delta(x,k)$$

Combinatorics for coloring over $\mathbb{Q}$

Let D be a dense linear order ($\simeq \mathbb{Q}$).
A function $f : D \to X$ is called *homogeneous* if $f^{-1}(x)$ is either dense or empty for every $x \in X$.

The shuffle principle

For any coloring $c : \mathbb{Q} \to [\![0, n]\!]$, there is $I \subseteq_{\mathrm{conv}} Q$ such that $c\big|_I$ is a shuffle.

- the key additional principle behind the usual inductive argument in [Carton, Colcombet, Puppis (2015)]

Shelah's additive Ramseyan theorem

Let M be a monoid. For every map $f : [\mathbb{Q}]^2 \to M$ such that $f(q,r)f(r,s) = f(q,s)$, there exists an interval $I \subseteq \mathbb{Q}$ and a finite partition into finitely many dense sets D_i of I such that f is constant over each $[D_i]^2$.

- the key additional principle behind the usual inductive argument in [Shelah (1975)]

Consider a formula $\varphi(u, x)$. $(u \in U^\omega, x \in X^\omega)$

$\rightsquigarrow$ Infinite 2-player game $\mathcal{G}_\varphi$ between P and O.

O	x_0		x_1		$\dots$	x_n		$\dots$
P		u_0		u_1			u_n	

P wins
$\iff$
$\varphi(u, x)$ holds

The Büchi-Landweber theorem

Consider a formula $\varphi(u, x)$. $(u \in U^\omega, x \in X^\omega)$

$\rightsquigarrow$ Infinite 2-player game $\mathcal{G}_\varphi$ between P and O.

O	x_0		x_1		$\dots$	x_n		$\dots$
P		u_0		u_1			u_n	

P wins
$\Longleftrightarrow$
$\varphi(u, x)$ holds

P-strategies $\simeq$ $X^+ \to U$
causal functions

O-strategies $\simeq$ $U^* \to X$
eager causal functions

The Büchi-Landweber theorem

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Theorem [Büchi-Landweber (1969)]

Suppose φ is MSO-definable. The game $\mathcal{G}_\varphi$ is determined:

- Either there exists a finite-state P-strategy $s_P(x)$ s.t. $\forall x \in X^\omega \quad \varphi(s_P(x), x)$
- Or there exists a finite-state O-strategy $s_O(u)$ s.t. $\forall u \in U^\omega \; \neg\varphi(u, s_O(u))$

The realizability notion for SMSO

Uniform non-deterministic automata

Tuples $\mathcal{A} = (Q, q_0, U, \delta_\mathcal{A}, \Omega_\mathcal{A}) : \Sigma$ where

- U a set of *moves* — $\simeq$ amount of non-determinism
- transition function $\delta_\mathcal{A} : \Sigma \times Q \times U \to Q$ — induces $\delta^*_\mathcal{A} : \Sigma^\omega \times U^\omega \to Q^\omega$
- $\Omega_\mathcal{A} \subseteq Q^\omega$ reasonable acceptance condition — (parity, Muller, ...)

- Same definable languages $\mathcal{L}(\mathcal{A}) = \{w \mid \exists u\ \delta^*_\mathcal{A}(w, u)\}$ — $U \simeq Q$

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Simulations $\mathcal{A} \Vdash f : \mathcal{B}$

Finite-state causal function $f : \Sigma^\omega \times U^\omega \to V^\omega$ such that

$$\forall w \in \Sigma^\omega \forall u \in U^\omega \qquad \delta^*_\mathcal{A}(w, u) \in \Omega_\mathcal{A} \quad \Rightarrow \quad \delta^*_\mathcal{A}(w, f(w, u)) \in \Omega_\mathcal{B}$$

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- If $\mathcal{A} \Vdash \mathcal{B}$, then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$
- Natural interpretation for $\exists$, $\wedge$ and $\neg$ for deterministic automata...

Alternating uniform automata

Define a notion of *alternating* uniform automata $(Q, q_0, U, X, \delta, \Omega)$

- sets of P-moves U and O-moves X
- $\delta : \Sigma \times Q \times U \times X \to Q$
- $w \in \mathcal{L}(\mathcal{A})$ iff P wins an *acceptance game*

Simulation game

$$
\begin{array}{c|ccccc}
 & (U & , X) & \multimap & (V, & Y) \\
\hline
 & & & \vdots & & \\
\mathrm{O} & u_n & & & & \\
\mathrm{P} & & & & v_n & \\
\mathrm{O} & & & & & y_n \\
\mathrm{P} & & x_n & & & \\
 & & & \vdots & &
\end{array}
$$

P wins iff

$\langle u, x \rangle$ P-winning $\Rightarrow$ $\langle v, y \rangle$ P-winning

- $X \simeq 1 \quad \rightsquigarrow$ non-deterministic uniform automata
- $U \simeq X \simeq 1 \quad \rightsquigarrow$ deterministic automata

trivial simulations