

Equational theories of algebraic operators on Weihrauch problems

Cécilia Pradic

⊇ lies and *revisionism*

and j.w.w. Eike Neumann, Arno Pauly & Ian Price

Dagstuhl meeting 25141

Setting for the motivation

Type 2 computability

Turing Machines with

- Input tape containing some $i \in 2^\omega$
- Write-and-go-right-only-output tape

- Natural setting to compute with infinite objects
(the “real” 2^ω is representable)

The category of represented spaces ReprSp

- Objects: (X, δ_X) where δ_X is a **partial** surjection $2^\omega \twoheadrightarrow X$
- Morphisms: maps $X \rightarrow X'$ with a type 2-computable witness

- Super nice: extensive, lcc, W/M -types
- (\cong subcategory of the modest sets in the Kleene-Vesley topos)

Weihrauch problems

Definition of Weihrauch problems as *containers*

A Weihrauch problem P is an internal family in ReprSp , i.e.

$$P : \text{positions}(P) \rightarrow \text{shape}(P)$$

- $\text{shape}(P)$ is the space of **questions**
- $\text{positions}(P)$ is the space of **answers**
- P links answers with the questions they are answering
- **Notation:** $P_i = P^{-1}(i)$

Examples:

- $C_{\mathbb{N}}$: “Given $p \in \mathbb{N}^{\mathbb{N}}$, find something not enumerated by p ”

$$\{(p, 1^n 0^\omega) \in \mathbb{N}^{\mathbb{N}} \mid n \notin \text{range}(p)\} = \text{positions}(C_{\mathbb{N}}) \xrightarrow{\pi_1} \text{shape}(C_{\mathbb{N}}) \subseteq \mathbb{N}^{\mathbb{N}}$$

- WKL_0 : “given an infinite binary tree, produce an infinite path”

Weihrauch reducibility

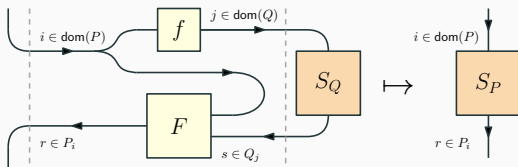
TL;DR: Turing reducibility, but

- adapted to type 2 computability
- reductions must make **exactly** one oracle call

Official definition

$P \leq_W Q$ if there are **computable**

$$f : \text{shape}(P) \rightarrow \text{shape}(Q) \quad \text{and} \quad F : \prod_{i \in \text{shape}(P)} (Q_{f(i)} \rightarrow P_i)$$



Reductions compose + Quotienting by $\equiv_W \rightsquigarrow$ **Weihrauch degrees**

The more general picture: container morphisms

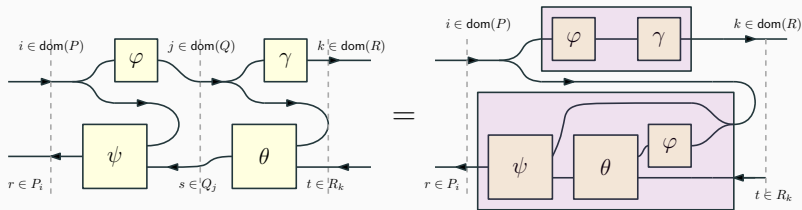
- Fix a category \mathcal{C} with **pullbacks**
- $\text{Cont}(\mathcal{C})$ has internal families in \mathcal{C} as objects

Official definition

A morphism $P \rightarrow Q$ in $\text{Cont}(\mathcal{C})$ is **a** pair (f, F) of

$$f : \text{shape}(P) \rightarrow \text{shape}(Q) \quad \text{and} \quad F : \prod_{i \in \text{shape}(P)} (Q_{f(i)} \rightarrow P_i)$$

(To make sense of what F is: requires pullbacks)



A Weihrach reduction $P \leq_W Q = \text{morphism } P \rightarrow Q \text{ in } \text{Cont}(\text{ReprSp})$ 5/23

Some functors on containers/Weihrauch problems

- Coproducts (joins) $+$:

$$\begin{aligned} \text{shape}(P + Q) &\cong \text{shape}(P) + \text{shape}(Q) & (P + Q)_{\text{in}_1(i)} &= P_i \\ & & (P + Q)_{\text{in}_2(j)} &= Q_j \end{aligned}$$

- Cartesian product \times : “given inputs for both, solve one”

$$\text{shape}(P \times Q) \cong \text{shape}(P) \times \text{shape}(Q) \quad (P \times Q)_{i,j} = P_i + Q_j$$

- Hadamard product \otimes : “solve both problems”

$$\text{shape}(P \otimes Q) \cong \text{shape}(P) \times \text{shape}(Q) \quad (P \otimes Q)_{i,j} = P_i \times Q_j$$

- **I**: $\text{shape}(\mathbf{I}) = \text{positions}(\mathbf{I}) = 1$

Composition, iterated composition

Sequential composition $Q \triangleright P$

- Implicitly: ability to make an oracle call to Q then P
- Explicitly: given an instance i of Q and a function that takes a solution of i to an instance of P , compute all relevant solutions

$$\begin{aligned}\text{shape}(Q \triangleright P) &\cong \sum_{i \in \text{shape}(Q)} (Q_i \rightarrow \text{shape}(P)) \\ (Q \triangleright P)_{i,f} &\cong \sum_{r \in Q_i} P_{f(r)}\end{aligned}$$

Iterated composition P^\triangleright

- Explicitly: computed as the least fixpoint of $X \mapsto \mathbf{I} + (P \triangleright X)$
- Implicitly: ability to make a finite but not fixed in advance number of oracle calls to P

Fixpoint of operators

least fixpoint	initial algebra	μ
greatest fixpoint	terminal coalgebra	ν

A very plausible conjecture (Folklore?)

If F is a fibred polynomial endofunctor over containers, the following exists:

- an initial algebra μF for F
- a terminal coalgebra νF for F
- a somewhat canonical bialgebra ζF sitting in-between

Examples:

- $P^\triangleright = \mu(X \mapsto \mathbf{I} + P \triangleright X)$
- $P^\otimes = \mu(X \mapsto \mathbf{I} + X \otimes P)$
- $P^{\otimes\infty} = \zeta(X \mapsto X \otimes P)$
- $P^{\triangleright\infty} = \zeta(X \mapsto P \triangleright X)$

Equational theory of the Weihrauch lattice

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, +, \times)$
(via the Medvedev degrees)
- Thus, $(\mathfrak{W}, +, \times) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

Equational theory of the Weihrauch lattice

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, +, \times)$
(via the Medvedev degrees)
- Thus, $(\mathfrak{W}, +, \times) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

Meta-question

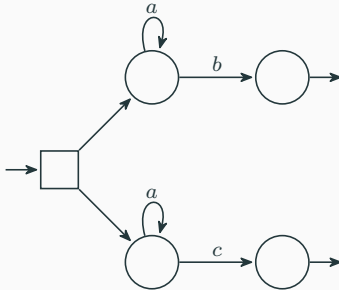
For a given signature, is there anything true in the Weihrauch degree that is not true for **all** (suitable) categories of containers?

Terms with composition and automata

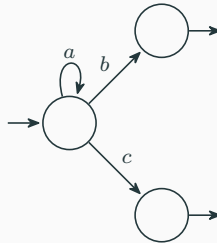
Starting observation

Terms over $0, \mathbf{I}, +, \triangleright, (-)^\triangleright =$ can be regarded as regular expressions.
(alphabet = the set of variables)

- Terms can be mapped to NFAs in a meaningful way
- Adding $\times =$ allowing alternating automata



$(a^\triangleright \triangleright b) \times (a^\triangleright \triangleright c)$



$a^\triangleright \triangleright (b + c)$

Universal validity and games

Given alternating automata \mathcal{A} and \mathcal{B} , we can define a game $\mathfrak{D}(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

Theorem

$(\mathfrak{W}, +, \times, \triangleright, (-)^\triangleright) \models t \leq u$ iff Duplicator wins in $\mathfrak{D}(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\mathfrak{D}(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating \mathcal{A} in parallel
(using \mathcal{B} exactly once)

Universal validity and games

Given alternating automata \mathcal{A} and \mathcal{B} , we can define a game $\mathfrak{D}(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

Theorem

$(\mathfrak{W}, +, \times, \triangleright, (-)^\triangleright) \models t \leq u$ iff Duplicator wins in $\mathfrak{D}(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\mathfrak{D}(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating \mathcal{A} in parallel
(using \mathcal{B} exactly once)

$\Rightarrow \mathcal{O}(|\mathcal{B}|2^{|\mathcal{A}|})$ positions

Universal validity and games

Given alternating automata \mathcal{A} and \mathcal{B} , we can define a game $\mathcal{D}(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

Theorem

$(\mathfrak{W}, +, \times, \triangleright, (-)^\triangleright) \models t \leq u$ iff Duplicator wins in $\mathcal{D}(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\mathcal{D}(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating \mathcal{A} in parallel
(using \mathcal{B} exactly once)

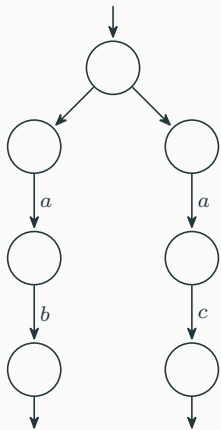
$\Rightarrow \mathcal{O}(|\mathcal{B}|2^{|\mathcal{A}|})$ positions

Corollary

“($\mathfrak{W}, \mathbf{I}, 0, +, \times, \triangleright, (-)^\triangleright$) $\models t \leq u$?” is decidable.

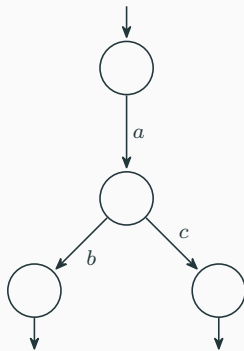
- Conjecture: this is PSPACE-complete.

A simple example of simulation and non-simulation



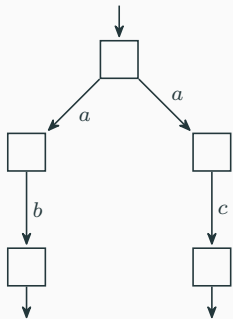
$(a \triangleright b) + (a \triangleright c)$

$<$



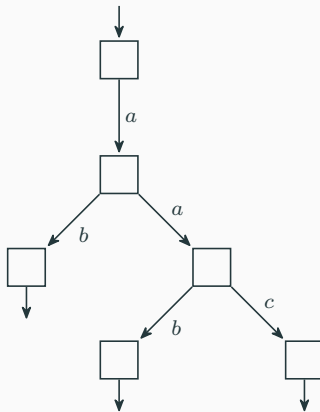
$a \triangleright (b + c)$

A simulation requiring several concurrent attempts



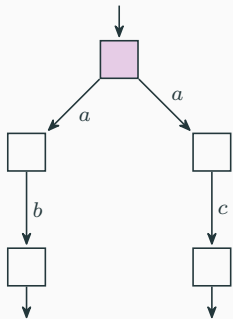
$(a \triangleright b) \times (a \triangleright c)$

$<$

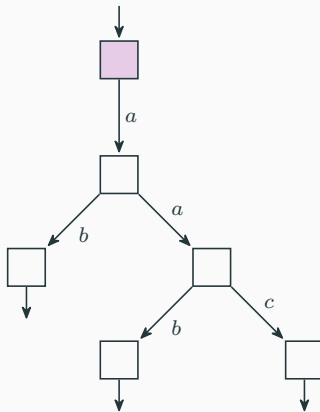


$a \triangleright ((a \triangleright (c \times b)) \times b)$

A simulation requiring several concurrent attempts



$$(a \triangleright b) \times (a \triangleright c)$$

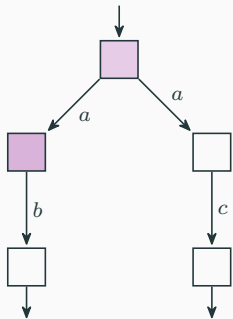


$$a \triangleright ((a \triangleright (c \times b)) \times b)$$

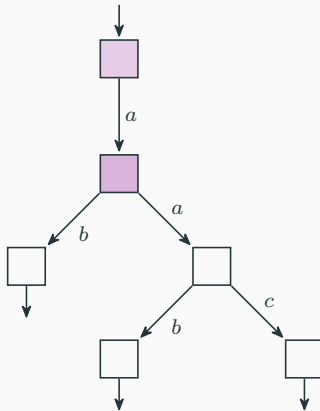
<

Let's play ?!

A simulation requiring several concurrent attempts



$$(a \triangleright b) \times (a \triangleright c)$$

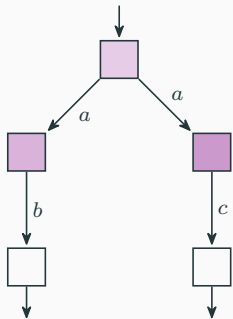


$$a \triangleright ((a \triangleright (c \times b)) \times b)$$

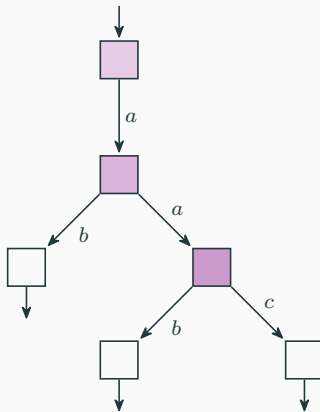
<

Let's play ?!

A simulation requiring several concurrent attempts



$$(a \triangleright b) \times (a \triangleright c)$$

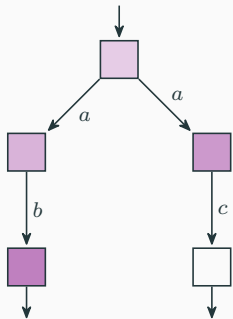


$$a \triangleright ((a \triangleright (c \times b)) \times b)$$

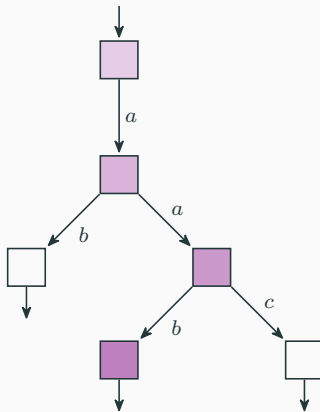
<

Let's play ?!

A simulation requiring several concurrent attempts



$$(a \triangleright b) \times (a \triangleright c)$$

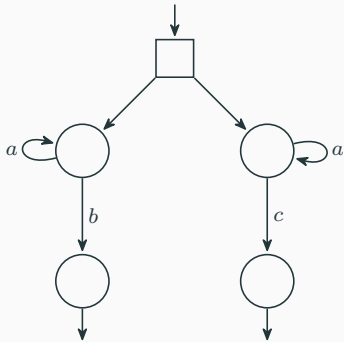


$$a \triangleright ((a \triangleright (c \times b)) \times b)$$

<

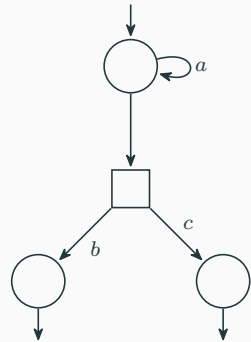
Let's play ?!

Another simulation requiring several concurrent attempts



$$(a^\triangleright \triangleright b) \times (a^\triangleright \triangleright c)$$

\equiv



$$a^\triangleright \triangleright (b \times c)$$

Induction principles for $(-)^{\triangleright}$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \triangleright x \leq x \quad \Rightarrow \quad x^{\triangleright} \leq x$$

Induction principles for $(-)^{\triangleright}$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \triangleright x \leq x \quad \Rightarrow \quad x^{\triangleright} \leq x$$

- Similar: an axiom of **left-handed Kleene algebras** (LKA)

$$y \triangleright x \leq x \quad \Rightarrow \quad y^{\triangleright} \triangleright x \leq x$$

Induction principles for $(-)^{\triangleright}$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \triangleright x \leq x \quad \Rightarrow \quad x^{\triangleright} \leq x$$

- Similar: an axiom of **left-handed Kleene algebras** (LKA)

$$y \triangleright x \leq x \quad \Rightarrow \quad y^{\triangleright} \triangleright x \leq x$$

- For \times , it seems like we sometimes need

$$(y \triangleright x) \times z \leq x \quad \Rightarrow \quad (y^{\triangleright} \triangleright x) \times z \leq x$$

(key example: $\mathbf{I} \leq a \times b$ implies $a^{\triangleright} \times b^{\triangleright} \leq (a \times b)^{\triangleright}$)

Induction principles for $(-)^{\triangleright}$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \triangleright x \leq x \quad \Rightarrow \quad x^{\triangleright} \leq x$$

- Similar: an axiom of **left-handed Kleene algebras** (LKA)

$$y \triangleright x \leq x \quad \Rightarrow \quad y^{\triangleright} \triangleright x \leq x$$

- For \times , it seems like we sometimes need

$$(y \triangleright x) \times z \leq x \quad \Rightarrow \quad (y^{\triangleright} \triangleright x) \times z \leq x$$

(key example: $\mathbf{I} \leq a \times b$ implies $a^{\triangleright} \times b^{\triangleright} \leq (a \times b)^{\triangleright}$)

Theorem

The above axioms are valid in the extended Weihrauch degrees.

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of $+$ over \star**

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of $+$ over \triangleright**
 - ★
 - i.e. it can be the case that $R \triangleright (P + Q) \not\equiv_w (R \triangleright P) + (R \triangleright Q)$

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of $+$ over**
 - ★
 - i.e. it can be the case that $R \triangleright (P + Q) \not\equiv_w (R \triangleright P) + (R \triangleright Q)$
 - why: LKA = language inclusions, but we want simulations

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of \triangleright over \times**
 - i.e. it can be the case that $R \triangleright (P + Q) \not\equiv_w (R \triangleright P) + (R \triangleright Q)$
 - why: LKA = language inclusions, but we want simulations
- The distributive lattice axioms **with units** $+$

$$\begin{aligned}(y \triangleright x) \times (z \triangleright x) &\leq (y \times z) \triangleright x \\(y \triangleright x) \times z &\leq y \triangleright (x \times z) \\(y \triangleright x) \times z \leq x &\Rightarrow (y \triangleright x) \times z \leq x \\1 \triangleright x = 1 &\quad 0 \triangleright x = 0\end{aligned}$$

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of $+$ over \triangleright**
 - i.e. it can be the case that $R \triangleright (P + Q) \not\equiv_w (R \triangleright P) + (R \triangleright Q)$
 - why: LKA = language inclusions, but we want simulations
- The distributive lattice axioms **with units $+$**

$$\begin{aligned}(y \triangleright x) \times (z \triangleright x) &\leq (y \times z) \triangleright x \\(y \triangleright x) \times z &\leq y \triangleright (x \times z) \\(y \triangleright x) \times z \leq x &\Rightarrow (y^\triangleright \triangleright x) \times z \leq x \\1 \triangleright x = 1 &\quad 0 \triangleright x = 0\end{aligned}$$

Theorem

Complete for the equational theory of $(\mathfrak{W}, \mathbf{I}, 0, 1, +, \times, \triangleright, (-)^\triangleright)$.

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of $+$ over \triangleright**
 - i.e. it can be the case that $R \triangleright (P + Q) \not\equiv_w (R \triangleright P) + (R \triangleright Q)$
 - why: LKA = language inclusions, but we want simulations
- The distributive lattice axioms **with units $+$**

$$\begin{aligned}(y \triangleright x) \times (z \triangleright x) &\leq (y \times z) \triangleright x \\(y \triangleright x) \times z &\leq y \triangleright (x \times z) \\(y \triangleright x) \times z \leq x &\Rightarrow (y^\triangleright \triangleright x) \times z \leq x \\1 \triangleright x = 1 &\quad 0 \triangleright x = 0\end{aligned}$$

Theorem

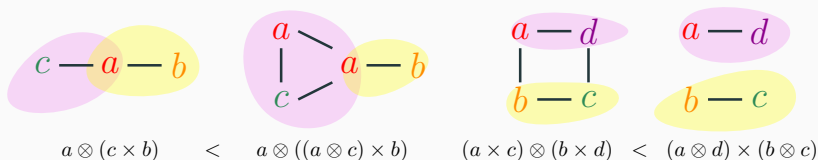
Complete for the equational theory of $(\mathfrak{W}, \mathbf{I}, 0, 1, +, \times, \triangleright, (-)^\triangleright)$.

Proof idea: \exists positional simulation strategies, induction on the syntax

How this started: the theory of \otimes, \times

A notion of combinatorial reduction between graphs

A reduction from (V_0, E_0, c_0) to (V_1, E_1, c_1) is a colour-preserving function $h : V_1 \rightarrow V_0$ such that the image of any maximal clique in (V_1, E_1) under h contains a maximal clique in (V_0, E_0) .



Combinatorial characterization (Neumann, Pauly, P.)

$(\mathfrak{W}, \times, \otimes) \models t \leq u$ iff there is a reduction from G_t to G_u .

As a result, deciding $(\mathfrak{W}, \times, \otimes) \models t \leq u$ is Σ_2^p -complete.

- Axiomatizing: harder!
- $+$, \triangleright and \otimes : opens the gates of hell (**concurrency theory**)

Conjectures!

(and related mess)

Extending the signature/the simulation game

- Enriching the signature with aforementioned μ = same thing with all finite alternating automata
- Then enriching the signature with ν = parity alternating automata
- Then enriching the signature with ζ (or $(-)^{\triangleright\infty}$) = runs of countable ordinal length
- Enriching with \otimes = going to higher-dimensional automata
 - Dealing with stuff that sounds like concurrency
 - Scarier to me!

Some englobing syntax for all signatures discussed here

$$\frac{x \in \Gamma \cup \Delta}{\Gamma; \Delta \vdash x}$$

$$\frac{\Gamma; \Delta \vdash t \quad \Gamma; \Delta \vdash u \quad \square \in \{\otimes, \times, +\}}{\Gamma; \Delta \vdash t \square u}$$

$$\frac{}{\Gamma; \Delta \vdash \mathbf{I}}$$

$$\frac{\Gamma; \cdot \vdash t \quad \Gamma; \Delta \vdash u}{\Gamma; \Delta \vdash t \triangleright u}$$

$$\frac{\Gamma; \Delta \vdash t \quad \Gamma; \cdot \vdash u}{\Gamma; \Delta \vdash t \blacktriangleright u}$$

$$\frac{\Gamma; \cdot \vdash t \quad \Gamma; \cdot \vdash u \quad -* \in \{-\circ, \Rightarrow\}}{\Gamma; \Delta \vdash t -* u}$$

$$\frac{\Gamma; \Delta, x \vdash t \quad \gamma \in \{\mu, \nu, \zeta\}}{\Gamma; \Delta \vdash \gamma x.t}$$

Another kind of questions

Conjecture(s)

For various signatures, true inequations in the slightly extended Weihrauch degrees are true in **all** categories of containers.

- Proofs of completeness = there exists messy enough problems to not create other true equations in Weihrauch degrees.
- When does that happen in a category \mathcal{C}

Conjecture: that's true when

For every $n \in \mathbb{N}$, there is

- an object A in \mathcal{C}
- a strong antichain of (regular?) subobjects $(V_i)_{i < n}$ of A
- with all $V_i \cup V_j$ are connected

Meta-question: motivations?

Would anyone care about similar results for other categories of containers?

A vaguer project

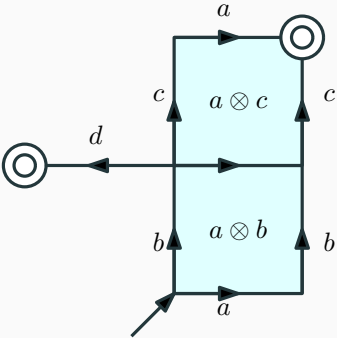
Inconvenient truths

- Weihrauch problems are the containers over **regular projective** represented spaces (subspaces of $\mathbb{N}^{\mathbb{N}}$) for which every question has an answer
- Containers over subspaces of Baire space are only **weakly** locally cartesian closed
- (and also have only weak (co)inductive types)
- It sounds unproblematic in practice because
 - The weak structure is good enough
 - (a systematic way of relating that = this is the category of regular projectives of represented spaces, which is a nice lccc)

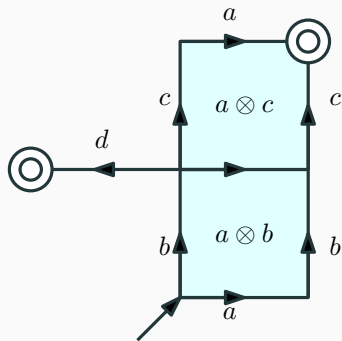
Question(s)

How do we transfer cleanly results about containers on a nice category \mathcal{C} with enough projectives to containers of the full subcategory of projectives?

Example of what's a higher-dimensional automaton



Example of what's a higher-dimensional automaton



(I dislike this HDA, I feel it is not nice enough to interpret in containers)