Weihrauch problems are containers. The equational theory of slightly extended Weihrauch degrees with composition.

Cécilia Pradic
$\supseteq$ j.w.w. Ian Price

Dagstuhl meeting 25131

Weihrauch problems

Definition

A Weihrauch problem P is given

- a set of instances $\mathsf{dom}(P) \subseteq \mathbb{N}^{\mathbb{N}}$
- for each $i \in \mathsf{dom}(P)$ a non-empty set of solutions $P_i \subseteq \mathbb{N}^{\mathbb{N}}$

Examples:

- $\mathsf{C}_{\mathbb{N}}$: "Given $p \in \mathbb{N}^{\mathbb{N}}$, find something not enumerated by p"

$\mathsf{dom}(\mathsf{C}_{\mathbb{N}}) = \{p \in \mathbb{N}^{\mathbb{N}} \mid \exists n\, n \notin \mathsf{range}(p)\}$ $\qquad \mathsf{C}_{\mathbb{N}}(p) = \{1^n 0^\omega \mid n \notin \mathsf{range}(p)\}$

- WKL_0: "given an infinite binary tree, produce an infinite path"

Weihrauch problems

Definition

A Weihrauch problem P is given

- a set of instances $\mathsf{dom}(P) \subseteq \mathbb{N}^{\mathbb{N}}$
- for each $i \in \mathsf{dom}(P)$ a non-empty set of solutions $P_i \subseteq \mathbb{N}^{\mathbb{N}}$

Examples:

- $\mathsf{C}_{\mathbb{N}}$: "Given $p \in \mathbb{N}^{\mathbb{N}}$, find something not enumerated by p"

$\mathsf{dom}(\mathsf{C}_{\mathbb{N}}) = \{p \in \mathbb{N}^{\mathbb{N}} \mid \exists n\, n \notin \mathsf{range}(p)\}$ $\qquad$ $\mathsf{C}_{\mathbb{N}}(p) = \{1^n 0^\omega \mid n \notin \mathsf{range}(p)\}$

- $\mathsf{WKL_0}$: "given an infinite binary tree, produce an infinite path"

Comparing the hardness of problems $\rightsquigarrow$ via a notion of reducibility

Weihrauch reducibility

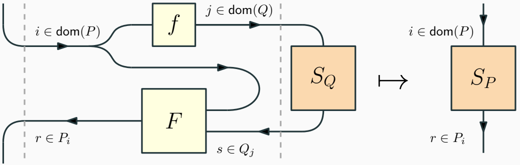
TL;DR: Turing reducibility, but

- adapted to type 2 computability
- reductions must make **exactly** one oracle call

Official definition

$P \leq_W Q$ if there are **computable**

$$f : \mathsf{dom}(P) \to \mathsf{dom}(Q) \quad \text{and} \quad F : \prod_{i \in \mathsf{dom}(P)} (Q_{f(i)} \to P_i)$$



Reductions compose + Quotienting by $\equiv_W$ $\rightsquigarrow$ **Weihrauch degrees**

Containers

Fix a category $\mathcal{C}$ with **pullbacks**

- minimal assumption to talk about "families of sets in $\mathcal{C}$"
- formally: morphisms $f : A \to B$ represents $(f^{-1}(a))_{a \in A}$

Definition

A container P is given by

- an object of shapes $\mathrm{shape}(P)$
- a family of solutions $(P_i)_{i \in \mathrm{shape}(P)}$

(formally a morphism $\mathrm{positions}(P) \to \mathrm{shape}(P)$)

Example (Weihrauch problems)

Call $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$ the category of subspaces of $\mathbb{N}^{\mathbb{N}}$ and computable maps between them.

All Weihrauch problems are containers.

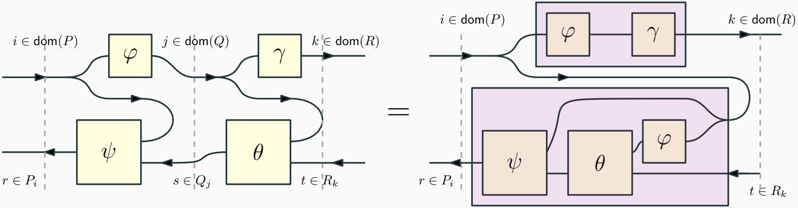
Container morphisms

Official definition

A morphism $P \to Q$ in $\mathsf{Cont}(\mathcal{C})$ is a pair (f, F) of

$$f : \text{shape}(P) \to \text{shape}(Q) \quad \text{and} \quad F : \prod_{i \in \text{shape}(P)} (Q_{f(i)} \to P_i)$$

(To make sense of what F is: requires pullbacks)



Containers over $\mathsf{pMod}(\mathcal{K}_2, \mathcal{K}_2^{\mathrm{rec}}) \approx$ Weihrauch problems

Not all containers in $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$ are Weihrauch problems

$$\mathsf{dom}(\top) = \{\bullet\} \qquad \top_\bullet = \emptyset$$

Call those containers where $P_i \ncong 0$ *answerable*

Contention/Theorem (P., Price)

Weihrauch problems/reducibility

$$\Longleftrightarrow$$

the fullsubcategory of answerable containers over $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$.

Containers over $\mathsf{pMod}(\mathcal{K}_2, \mathcal{K}_2^{\mathrm{rec}}) \approx$ Weihrauch problems

Not all containers in $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$ are Weihrauch problems

$$\mathsf{dom}(\top) = \{\bullet\} \qquad \top_\bullet = \emptyset$$

Call those containers where $P_i \ncong 0$ *answerable*

Contention/Theorem (P., Price)

Weihrauch problems/reducibility

$$\Longleftrightarrow$$

the fullsubcategory of answerable containers over $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$.

(Theorem: the degree structures are isomorphic)

Containers over $\mathsf{pMod}(\mathcal{K}_2, \mathcal{K}_2^{\mathrm{rec}}) \approx$ Weihrauch problems

Not all containers in $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$ are Weihrauch problems

$$\mathsf{dom}(\top) = \{\bullet\} \qquad \top_\bullet = \emptyset$$

Call those containers where $P_i \not\cong 0$ *answerable*

Contention/Theorem (P., Price)

Weihrauch problems/reducibility

$$\Longleftrightarrow$$

the fullsubcategory of answerable containers over $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$.

(Theorem: the degree structures are isomorphic)

- For structural stuff, answerability is annoying
- ~~answerable~~ = **slightly extended Weihrauch problems**

(terminology suggestions welcome)

Extended Weihrauch problems

- Assume AC for this slide
- $\mathsf{pAsm}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$ = multirepresented subspaces of some $\nabla(X) \times \mathbb{N}^{\mathbb{N}}$

Theorem (P., Price)

The degree structure of containers over $\mathsf{pAsm}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$ is the same as extended Weihrauch degrees.

- This says nothing about instance reducibility in general.

Other things we know how to do

- Trivially: continuous/generalized W reducibility
- With some work: strong reducibility

Point? (not sure)

Seen in the container literature

$$\sqcap \qquad \sqcup \qquad \star \qquad \times \qquad (-)^{\diamond} \qquad \to \qquad / \qquad \Rightarrow$$

- Assuming we are working in a elcc with (co)inductive types
- Sadly not **quite** true for $\mathsf{pAsm}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2))$
- Let's pretend it is :) for now

Point? (not sure)

Seen in the container literature

$$\sqcap \qquad \sqcup \qquad \star \qquad \times \qquad (-)^{\diamond} \qquad \to \qquad / \qquad \Rightarrow$$

- Assuming we are working in a elcc with (co)inductive types
- Sadly not **quite** true for $\mathsf{pAsm}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2))$
- Let's pretend it is :) for now
  Maths is easier when assuming something patently false!

Point? (not sure)

Seen in the container literature

$$\sqcap \qquad \sqcup \qquad \star \qquad \times \qquad (-)^{\diamond} \qquad \rightarrow \qquad / \qquad \Rightarrow$$

- Assuming we are working in a elcc with (co)inductive types
- Sadly not **quite** true for $\mathsf{pAsm}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2))$
- Let's pretend it is :) for now
  Maths is easier when assuming something patently false!

- A lot of work on type theory and containers (von Glehn & Moss 18)
  - Another way to link linear arithmetic/Weihrauch reducibility?
    (other than (Uftring 21); I'm not optimistic atm)

Some functors on containers (operators on Weihrauch problems)

Many natural operators over Weihrauch problems/degrees:

- Coproducts (joins) $\sqcup$:

$$\mathsf{dom}(P \sqcup Q) \cong \mathsf{dom}(P) + \mathsf{dom}(Q) \qquad \begin{array}{rcl} (P \sqcup Q)_{\mathsf{in}_1(i)} & = & P_i \\ (P \sqcup Q)_{\mathsf{in}_2(j)} & = & Q_j \end{array}$$

- Meets $\sqcap$: “given inputs for both, solve one”

$$\mathsf{dom}(P \sqcap Q) \cong \mathsf{dom}(P) \times \mathsf{dom}(Q) \quad (P \sqcap Q)_{i,j} = P_i + Q_j$$

- Products $\times$: “solve both problems”

$$\mathsf{dom}(P \times Q) \cong \mathsf{dom}(P) \times \mathsf{dom}(Q) \quad (P \times Q)_{i,j} = P_i \times Q_j$$

- 1: “there is a computable instance which has a computable solution”

Fixpoint of operators

least fixpoint	initial algebra	μ
greatest fixpoint	terminal coalgebra	ν

A very plausible conjecture (Folklore?)

If F is a fibred polynomial endofunctor over containers, the following exists:

- an initial algebra μF for F
- a terminal coalgebra νF for F
- a somewhat canonical bialgebra ζF sitting in-between

Examples:

- $P^{\diamond} = \mu(X \mapsto 1 \sqcup X \star P)$
- $P^{*} = \mu(X \mapsto 1 \sqcup X \times P)$
- $\widehat{P} = \zeta(X \mapsto X \times P)$
- $P^{\infty} = \zeta(X \mapsto X \star P)$

Abstract nonsense over!
(Talk 2)

Equational theory of the s.e. Weihrauch lattice

- The (s.e.) Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, \sqcup, \sqcap)$

  (via the Medvedev degrees)
- Thus, $(\mathfrak{W}, \sqcup, \sqcap) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

Equational theory of the s.e. Weihrauch lattice

- The (s.e.) Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, \sqcup, \sqcap)$

  (via the Medvedev degrees)
- Thus, $(\mathfrak{W}, \sqcup, \sqcap) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

Meta-question

For a given signature, is there anything true in the s.e. Weihrauch degree that is not true for **all** (suitable) categories of containers?

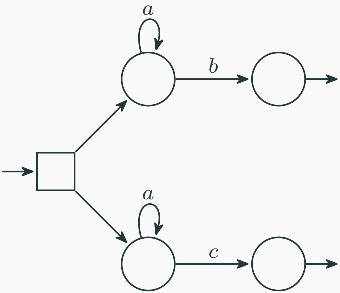
Terms with composition and automata

Starting observation

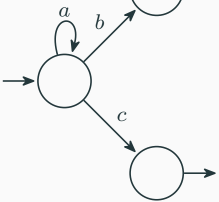
Terms over $0, 1, \sqcup, \star, (-)^\diamond$ = can be regarded as regular expressions.

(alphabet = the set of variables)

- Terms can be mapped to NFAs in a meaningful way
- Adding $\sqcap$ = allowing alternating automata



$(b \star a^\diamond) \sqcap (c \star a^\diamond)$



$(b \sqcup c) \star a^\diamond$

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\Game(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

Theorem

$(\mathfrak{W}, \sqcup, \sqcap, \star, (-)^{\diamond}) \models t \leq u$ iff Duplicator wins in $\Game(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\Game(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel

(using $\mathcal{B}$ exactly once)

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\Game(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

Theorem

$(\mathfrak{W}, \sqcup, \sqcap, \star, (-)^\diamond) \models t \leq u$ iff Duplicator wins in $\Game(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\Game(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel

(using $\mathcal{B}$ exactly once)

$\Rightarrow$ $\mathcal{O}(|\mathcal{B}|2^{|\mathcal{A}|})$ positions

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\eth(\mathcal{A},\mathcal{B})$ that captures a notion of simulation such that

Theorem

$(\mathfrak{W}, \sqcup, \sqcap, \star, (-)^\diamond) \models t \leq u$ iff Duplicator wins in $\eth(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\eth(\mathcal{A},\mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel

(using $\mathcal{B}$ exactly once)

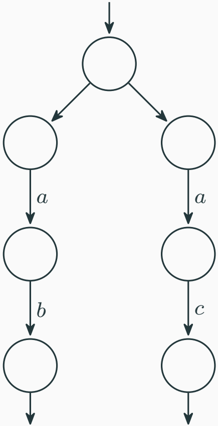
$\Rightarrow$ $\mathcal{O}(|\mathcal{B}|2^{|\mathcal{A}|})$ positions

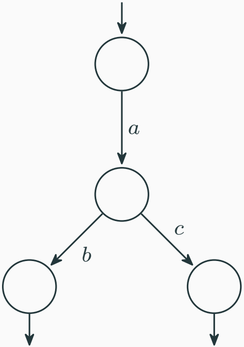
Corollary

The equational theory of "$(\mathfrak{W}, 1, 0, \sqcup, \sqcap, \star, (-)^\diamond) \models t \leq u$?" is decidable.

- Conjecture: this is PSPACE-complete.

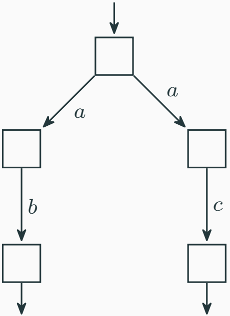
A simple example of simulation and non-simulation

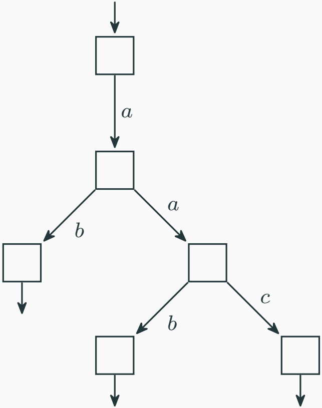




$$(b \star a) \sqcup (c \star a) \quad < \quad (b \sqcup c) \star a$$

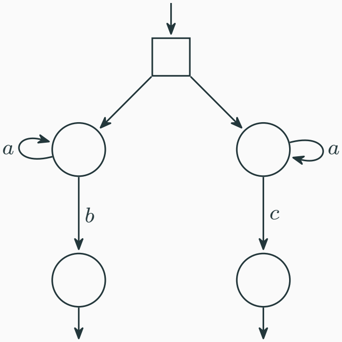
A simulation requiring several concurrent attempts

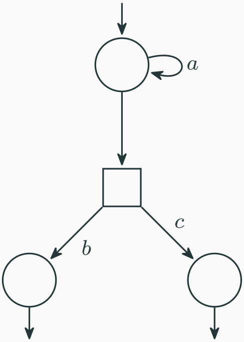




$$(b \star a) \sqcap (c \star a) \quad < \quad (((c \sqcap b) \star a) \sqcap b) \star a$$

Another simulation requiring several concurrent attempts





$$(b \star a^{\diamond}) \sqcap (c \star a^{\diamond}) \qquad \equiv \qquad (b \sqcap c) \star a^{\diamond}$$

Induction principles for $(-)^\diamond$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \star x \leq x \qquad \Rightarrow \qquad x^\diamond \leq x$$

Induction principles for $(-)^\diamond$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \star x \leq x \qquad \Rightarrow \qquad x^\diamond \leq x$$

- Similar: an axiom of **right-handed Kleene algebras** (RKA)

$$x \star y \leq x \qquad \Rightarrow \qquad x \star y^\diamond \leq x$$

Induction principles for $(-)^\diamond$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \star x \leq x \qquad \Rightarrow \qquad x^\diamond \leq x$$

- Similar: an axiom of **right-handed Kleene algebras** (RKA)

$$x \star y \leq x \qquad \Rightarrow \qquad x \star y^\diamond \leq x$$

- For $\sqcap$, it seems like we sometimes need

$$(x \star y) \sqcap z \leq x \qquad \Rightarrow \qquad (x \star y^\diamond) \sqcap z \leq x$$

(key example: $a^\diamond \sqcap b^\diamond \leq (a \sqcap b)^\diamond$)

Induction principles for $(-)^\diamond$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \star x \leq x \qquad \Rightarrow \qquad x^\diamond \leq x$$

- Similar: an axiom of **right-handed Kleene algebras** (RKA)

$$x \star y \leq x \qquad \Rightarrow \qquad x \star y^\diamond \leq x$$

- For $\sqcap$, it seems like we sometimes need

$$(x \star y) \sqcap z \leq x \qquad \Rightarrow \qquad (x \star y^\diamond) \sqcap z \leq x$$

(key example: $a^\diamond \sqcap b^\diamond \leq (a \sqcap b)^\diamond$)

Theorem

The above axioms are valid in the extended Weihrauch degrees.

Completeness

Candidate axiomatization of inequations

- All the axioms of RKA **minus left-distributivity of $\sqcup$ over $\star$**

Completeness

Candidate axiomatization of inequations

- All the axioms of RKA **minus left-distributivity of $\sqcup$ over $\star$**
  - i.e. it can be the case that $(P \sqcup Q) \star R \not\equiv_{\mathrm{W}} (P \star R) \sqcup (Q \star R)$

Completeness

Candidate axiomatization of inequations

- All the axioms of RKA **minus left-distributivity of $\sqcup$ over $\star$**
  - i.e. it can be the case that $(P \sqcup Q) \star R \not\equiv_{\mathrm{W}} (P \star R) \sqcup (Q \star R)$
  - why: RKA = language inclusions, but we want simulations

Completeness

Candidate axiomatization of inequations

- All the axioms of RKA **minus left-distributivity of $\sqcup$ over $\star$**
  - i.e. it can be the case that $(P \sqcup Q) \star R \not\equiv_{\mathrm{W}} (P \star R) \sqcup (Q \star R)$
  - why: RKA = language inclusions, but we want simulations
- The distributive lattice axioms **with units** +

$$
\begin{array}{rcl}
(x \star y) \sqcap (x \star z) & \leq & x \star (y \sqcap z) \\
(x \star y) \sqcap z & \leq & (x \sqcap z) \star y \\
(x \star y) \sqcap z \leq x & \Rightarrow & (x \star y^{\diamond}) \sqcap z \leq x \\
x \star \top = \top & & \\
x \star 0 = 0 & &
\end{array}
$$

Completeness

Candidate axiomatization of inequations

- All the axioms of RKA **minus left-distributivity of $\sqcup$ over $\star$**
  - i.e. it can be the case that $(P \sqcup Q) \star R \not\equiv_{\mathrm{W}} (P \star R) \sqcup (Q \star R)$
  - why: RKA = language inclusions, but we want simulations
- The distributive lattice axioms **with units** +

$$
\begin{array}{rcl}
(x \star y) \sqcap (x \star z) & \leq & x \star (y \sqcap z) \\
(x \star y) \sqcap z & \leq & (x \sqcap z) \star y \\
(x \star y) \sqcap z \leq x & \Rightarrow & (x \star y^{\diamond}) \sqcap z \leq x \\
x \star \top = \top & & \\
x \star 0 = 0 & &
\end{array}
$$

Theorem

Complete for the equational theory of $(se\mathfrak{W}, 1, 0, \top, \sqcup, \sqcap, \star, (-)^{\diamond})$.

Completeness

Candidate axiomatization of inequations

- All the axioms of RKA **minus left-distributivity of $\sqcup$ over $\star$**
  - i.e. it can be the case that $(P \sqcup Q) \star R \not\equiv_{\mathrm{W}} (P \star R) \sqcup (Q \star R)$
  - why: RKA = language inclusions, but we want simulations
- The distributive lattice axioms **with units** +

$$
\begin{aligned}
(x \star y) \sqcap (x \star z) \quad &\leq \quad x \star (y \sqcap z) \\
(x \star y) \sqcap z \quad &\leq \quad (x \sqcap z) \star y \\
(x \star y) \sqcap z \leq x \quad &\Rightarrow \quad (x \star y^{\diamond}) \sqcap z \leq x \\
x \star \top = \top & \\
x \star 0 = 0 &
\end{aligned}
$$

Theorem

Complete for the equational theory of $(se\mathfrak{W}, 1, 0, \top, \sqcup, \sqcap, \star, (-)^{\diamond})$.

Proof idea: $\exists$ positional simulation strategies, induction on the syntax

Conjectures!
(and related mess)

Extending the signature/the simulation game

- Enriching the signature with aforementioned μ = same thing with all finite alternating automata
- Then enriching the signature with ν = parity alternating automata
- Then enriching the signature with ζ (or $(-)^\infty$) = runs of countable ordinal length
- Enriching with $\times$ = going to higher-dimensional automata
  - Dealing with stuff that sounds like concurrency
  - Scarier to me!

Some englobing syntax for all signatures discussed here

$$\frac{x \in \Gamma \cup \Delta}{\Gamma; \Delta \vdash x} \qquad \frac{\Gamma; \Delta \vdash t \qquad \Gamma; \Delta \vdash u \qquad \square \in \{\times, \sqcap, \sqcup\}}{\Gamma; \Delta \vdash t \mathbin{\square} u}$$

$$\frac{}{\Gamma; \Delta \vdash 1} \qquad \frac{\Gamma; \cdot \vdash t \qquad \Gamma; \Delta \vdash u}{\Gamma; \Delta \vdash t \star u}$$

$$\frac{\Gamma; \Delta \vdash t \qquad \Gamma; \cdot \vdash u}{\Gamma; \Delta \vdash u \to t} \qquad \frac{\Gamma; \cdot \vdash t \qquad \Gamma; \cdot \vdash u \qquad \twoheadrightarrow \in \{\multimap, \Rightarrow\}}{\Gamma; \Delta \vdash t \twoheadrightarrow u}$$

$$\frac{\Gamma; \Delta, x \vdash t \qquad \gamma \in \{\mu, \nu, \zeta\}}{\Gamma; \Delta \vdash \gamma x.t}$$

Another kind of questions

Conjecture(s)

For various signatures, true inequations in the slightly extended Weihrauch degrees are true in **all** categories of containers.

- Proofs of completeness = there exists messy enough problems to not create other true equations in Weihrauch degrees.
- When does that happen in a category $\mathcal{C}$

Conjecture: that's true when

For every $n \in \mathbb{N}$, there is

- an object A in $\mathcal{C}$
- a strong antichain of (regular?) subobjects $(V_i)_{i<n}$ of A
- with all $V_i \cup V_j$ are connected

A vaguer project

Containers over subspaces of Baire space are only **weakly** locally cartesian closed

- (and also have only weak (co)inductive types)
- It sounds unproblematic in practice because
  - The weak structure is good enough
  - (a systematic way of relating that = this is the category of regular projectives of represented spaces, which is a nice lccc)

Question(s)

How do we transfer cleanly results about containers on a nice category $\mathcal{C}$ with enough projectives to containers of the full subcategory of projectives?

Example of what's a higher-dimensional automaton