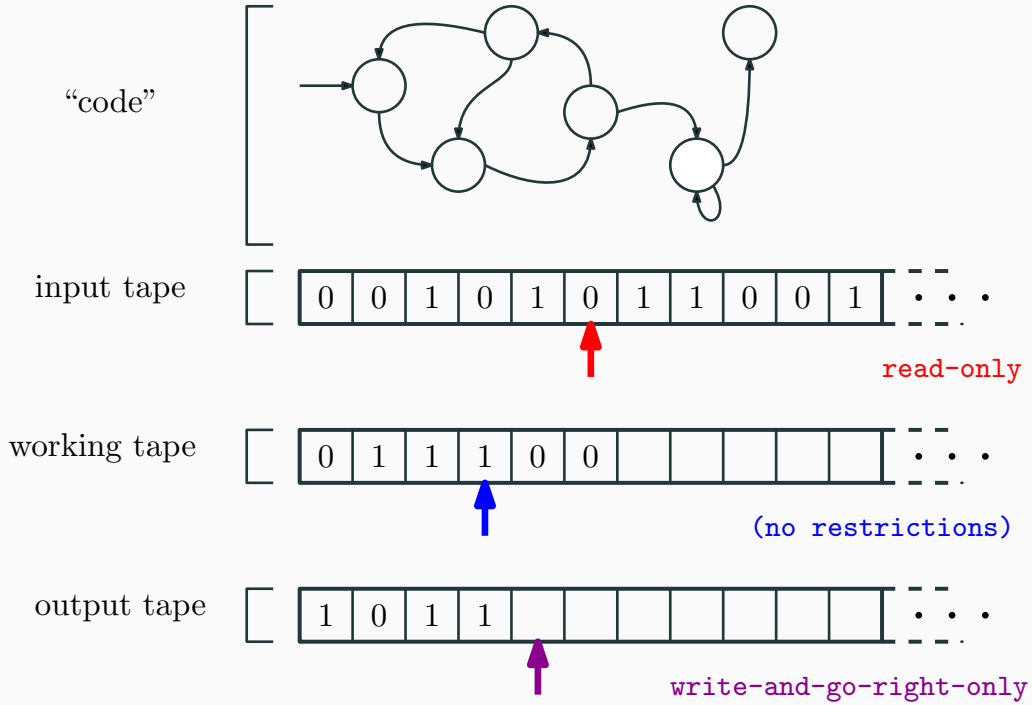
Automata, types and the structure of Weihrauch reducibility

Cécilia Pradic
⊇ j.w.w.w Eike Neumann, Arno Pauly, Ian Price, Manlio Valenti, ...

October 15th – **chocola**

∃ type-2 computability

The natural notion of computability for $2^{\mathbb{N}} \rightharpoonup 2^{\mathbb{N}}$

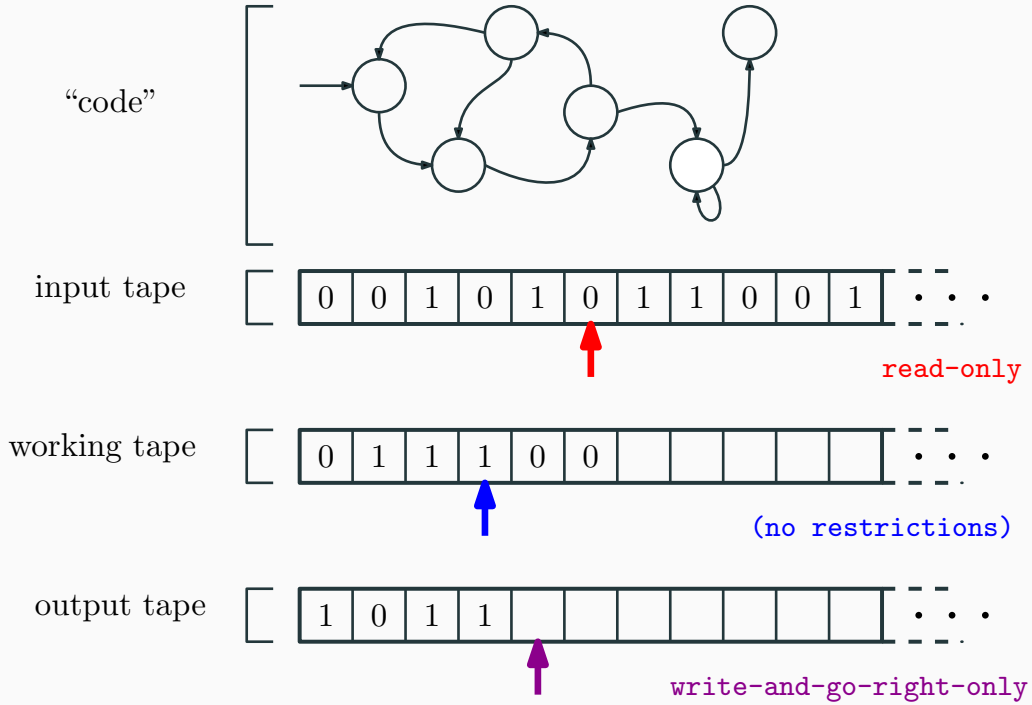


Formally, $f : 2^{\mathbb{N}} \rightharpoonup 2^{\mathbb{N}}$ computable

$\iff \exists \sqsubseteq$-monotone computable $\widehat{f} : 2^* \to 2^*$ s.t. $f(p) = \lim\limits_{n \to +\infty} \widehat{f}(p[n])$

∃ type-2 computability

The natural notion of computability for $2^{\mathbb{N}} \rightharpoonup 2^{\mathbb{N}}$



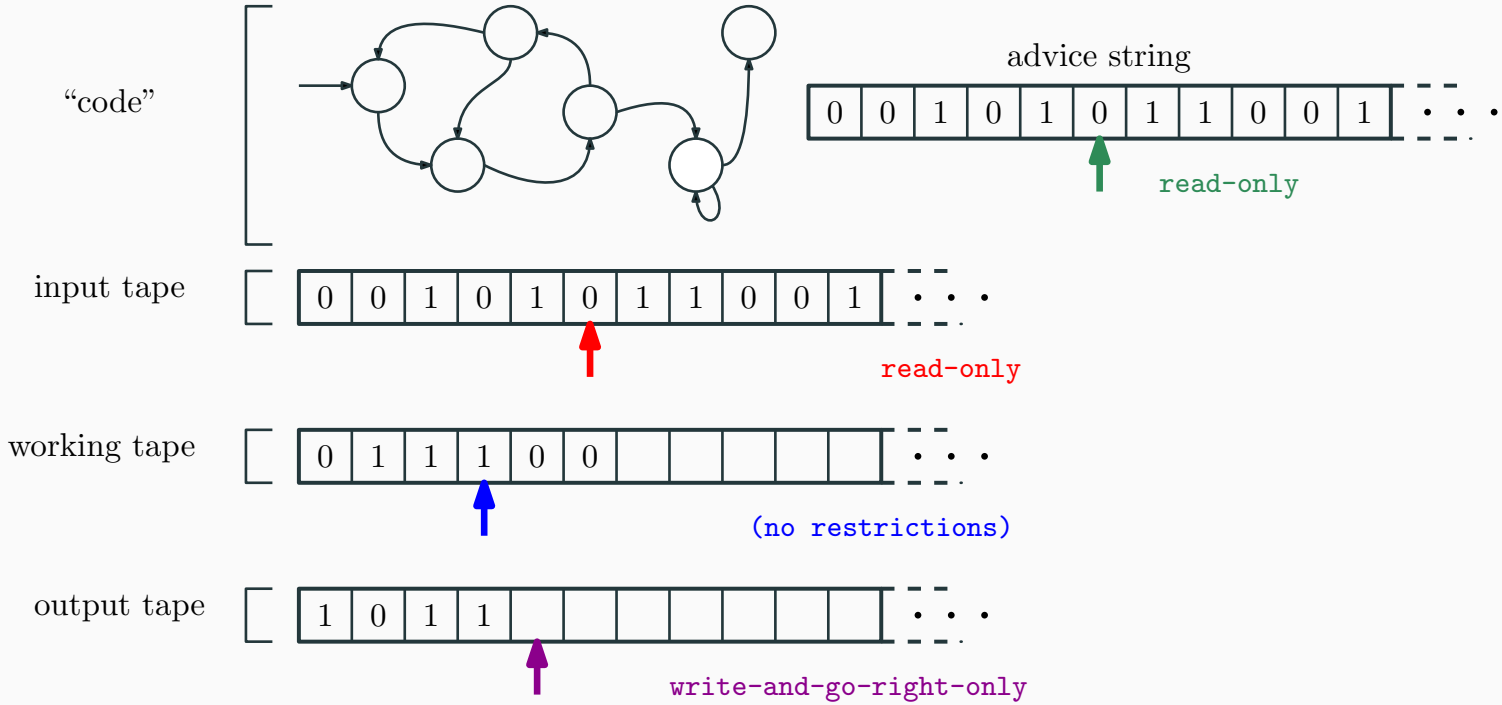
Formally, $f : 2^{\mathbb{N}} \rightharpoonup 2^{\mathbb{N}}$ computable

$\iff \exists \sqsubseteq$-monotone computable $\widehat{f} : 2^* \to 2^*$ s.t. $f(p) = \lim\limits_{n \to +\infty} \widehat{f}(p[n])$

$\iff \exists e \in \mathbb{N}.\, \forall n \in \mathbb{N}. \forall A \in \mathrm{dom}(f).\, \Phi_e^A(n) = f(A)(n)$

Continuity = type-2 computability relativized

Up to a type-1 oracle, all continuous functions are computable.



Formally, $f : 2^{\mathbb{N}} \rightharpoonup 2^{\mathbb{N}}$ continuous

$\iff \exists \sqsubseteq$-monotone $\widehat{f} : 2^* \to 2^*$ s.t. $f(p) = \lim\limits_{n \to +\infty} \widehat{f}(p[n])$

$\iff \exists e \in \mathbb{N}. \exists E \in 2^{\mathbb{N}}\ \forall n \in \mathbb{N}. \forall A \in \mathrm{dom}(f).\ \Phi_e^{E \oplus A}(n) = f(A)(n)$

Represented spaces

The category of represented spaces ReprSp

- Objects: (X, δ_X) where δ_X is a **partial** surjection $2^\omega \twoheadrightarrow X$
- Morphisms: maps $X \to X'$ with a type 2-computable witness

- Super nice: extensive, locally cartesian closed, (co)inductive types

Examples:

- Standard spaces like $\mathbb{N}$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$

$$\delta_{2^{\mathbb{N}}} = \mathrm{id}_{2^{\mathbb{N}}} \qquad \delta_{\mathbb{N}}(0^n 1 p) = n \qquad \delta_{\mathbb{N}^{\mathbb{N}}}(0^{p(0)} 1 0^{p(1)} 1 \ldots) = p$$

- $\mathbb{R}$ via fast Cauchy sequences $\mathbb{Q}^{\mathbb{N}}$ (convergence rate 2^{-n}).
- More generally: all subspaces of (quasi/co)Polish spaces.

A setting for effective topology (1/2)

We can adapt basic definitions from synthetic topology.

The Sierpiński space $\mathbb{S}$

As a represented space, define $\delta_{\mathbb{S}} : 2^{\mathbb{N}} \to \{\bot, \top\}$ by

$$\delta_{\mathbb{S}}(x) = \bot \qquad \iff \qquad x = 0^\omega$$

The spaces of open and closed subsets of X

We set $\mathcal{O}(X) = \mathcal{A}(X) = \mathbb{S}^X$, dual interpretations of $\in$

$$x \in U \in \mathcal{O}(X) \iff U(x) = \top \qquad x \in F \in \mathcal{A}(X) \iff F(x) = \bot$$

- Code for an open = eventually we notice if we are in
- Computably open = $\in$ is semi-computable
- $\bigcup : \mathcal{O}(\mathcal{O}(X)) \to \mathcal{O}(X)$, $\cup$, $\cap$, computable, ...

A setting for effective topology (2/2)

The space of compact subsets $\mathcal{K}(X)$

A subspace $\mathcal{K}(X) \subseteq \mathcal{O}(\mathcal{O}(X))$; concretely $\delta_{\mathcal{K}(X)}(p) = A \subseteq X$ iff

- we have $f \in \mathbb{S}^{\mathbb{S}^X}$ with $\delta_{\mathcal{O}(\mathcal{O}(X))}(p) = f$
- for all open $U \in \mathcal{O}(X)$, f detects if U covers A

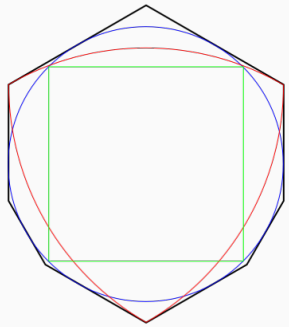
$$A \subseteq U \iff f(U) = \top$$

- X is computably compact iff $\forall_X : \mathcal{O}(X) \to \mathbb{S}$ is computable
- $[0,1]^\omega$, $2^{\mathbb{N}}$ computably compact...
- There exists a dual notion to compactness which is trivial classically, but relevant computably.

An application: Universal Lebesgue covering Problem

Theorem (Pauly & Ruggles '23)

The area of the smallest convex shape in $\mathbb{R}^2$ that cover all shapes of diameter 1 **is a computable real**.



Proof ideas:

- Show separately it is an upper/lower-semicomputable real
- $\inf U$ is upper-semi computable for a computable $U \in \mathcal{O}(\mathbb{R})$
- Use effective compactness and overtness over various things: compact subsets of $[0,1]^2$, $\mathrm{GL}_2([-2,2])$, ...

Non-computable, yet interesting tasks

- For given $x, y \in X$, do we have $x = y$ or not? $(=_X)$
- Given $p \in 2^{\mathbb{N}}$, does $p = 0^\omega$ or not? (LPO)
- Given $x, y \in \mathbb{R}$, do we have $x \leq y$ or $y \leq x$? (LLPO)
- Given a non-empty $A \in \mathcal{A}(X)$, find $x \in A$. (C_X)
- From $a, b \in \mathbb{R}$, find x such that $ax = b$. $(\mathsf{urbDiv}, \ldots)$
- Given a colouring $[\mathbb{N}]^k \to n$, produce an homogeneous set. (RT^k_n)
- Find paths in various classes of ω-branching trees. $(\mathsf{WKL}_0, \ldots)$
- $\ldots$

Motivation for Weihrauch complexity

Compare the relative hardness of these tasks.

Weihrauch problems

Partial Weihrauch problems, formally

Families $(P_i)_{i \in \mathrm{dom}(P)}$ with $\mathrm{dom}(P) \subseteq 2^{\mathbb{N}}$ and $P_i \subseteq 2^{\mathbb{N}}$ for every i.

($\simeq$ internal families in regular projective represented spaces)

- $\mathrm{dom}(P)$ = set of **questions**
- P_i = set of **answers** to the question i
- Totality = $\forall i \in \mathrm{dom}(P).\ P_i \neq \emptyset$

Weihrauch problems

Partial Weihrauch problems, formally

Families $(P_i)_{i\in\mathrm{dom}(P)}$ with $\mathrm{dom}(P) \subseteq 2^{\mathbb{N}}$ and $P_i \subseteq 2^{\mathbb{N}}$ for every i.

($\simeq$ internal families in regular projective represented spaces)

- $\mathrm{dom}(P)$ = set of **questions**
- P_i = set of **answers** to the question i
- Totality = $\forall i \in \mathrm{dom}(P).\ P_i \neq \emptyset$

Examples:

- LPO: "Given $p \in 2^{\mathbb{N}}$, return 0 iff $p = 0^\omega$":

  $\mathsf{LPO}_{0^\omega} = \{0^\omega\}$ $\mathsf{LPO}_p = \{10^\omega\}$ otherwise

Weihrauch problems

Partial Weihrauch problems, formally

Families $(P_i)_{i\in\mathrm{dom}(P)}$ with $\mathrm{dom}(P) \subseteq 2^{\mathbb{N}}$ and $P_i \subseteq 2^{\mathbb{N}}$ for every i.

($\simeq$ internal families in regular projective represented spaces)

- $\mathrm{dom}(P)$ = set of **questions**
- P_i = set of **answers** to the question i
- Totality = $\forall i \in \mathrm{dom}(P).\ P_i \neq \emptyset$

Examples:

- LPO: "Given $p \in 2^{\mathbb{N}}$, return 0 iff $p = 0^\omega$":

  $\mathsf{LPO}_{0^\omega} = \{0^\omega\} = \delta^{-1}_{\{0,1\}}(0)$ $\quad$ $\mathsf{LPO}_p = \{10^\omega\} = \delta^{-1}_{\{0,1\}}(1)$ $\quad$ otherwise

Weihrauch problems

Partial Weihrauch problems, formally

Families $(P_i)_{i \in \mathrm{dom}(P)}$ with $\mathrm{dom}(P) \subseteq 2^{\mathbb{N}}$ and $P_i \subseteq 2^{\mathbb{N}}$ for every i.

($\simeq$ internal families in regular projective represented spaces)

- $\mathrm{dom}(P)$ = set of **questions**
- P_i = set of **answers** to the question i
- Totality = $\forall i \in \mathrm{dom}(P).\ P_i \neq \emptyset$

Examples:

- LPO: "Given $p \in 2^{\mathbb{N}}$, return 0 iff $p = 0^\omega$":

  $\mathsf{LPO}_{0^\omega} = \{0^\omega\} = \delta^{-1}_{\{0,1\}}(0)$ $\quad$ $\mathsf{LPO}_p = \{10^\omega\} = \delta^{-1}_{\{0,1\}}(1)$ $\quad$ otherwise

- $\mathsf{C}_{\mathbb{N}}$, up to messing with representations for $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$:

$$\begin{aligned} \mathrm{dom}(\mathsf{C}_{\mathbb{N}}) &= \{p \in \mathbb{N}^{\mathbb{N}} \mid \exists n \in \mathbb{N}.\ n \notin \mathsf{range}(p)\} \\ (\mathsf{C}_{\mathbb{N}})_p &= \{n \in \mathbb{N} \mid n \notin \mathsf{range}(p)\} \end{aligned}$$

Weihrauch reducibility

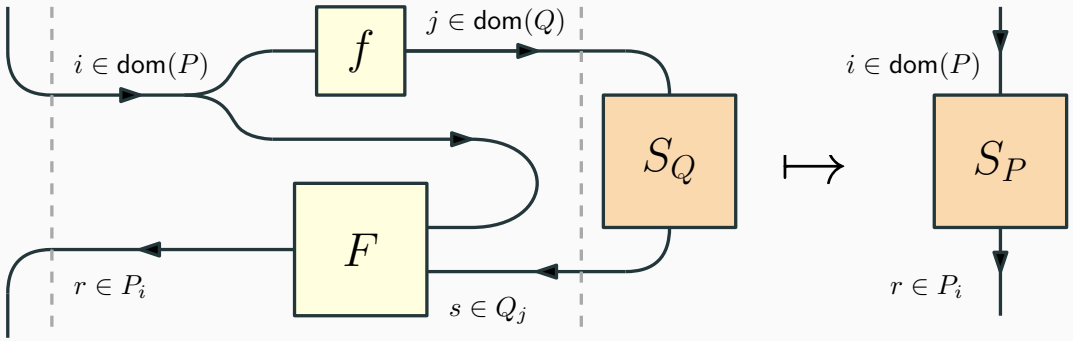
TL;DR: almost like Turing reducibility, but

- adapted to type 2 computability
- reductions must make **exactly** one oracle call
- (and Weihrauch problems are "non-deterministic")

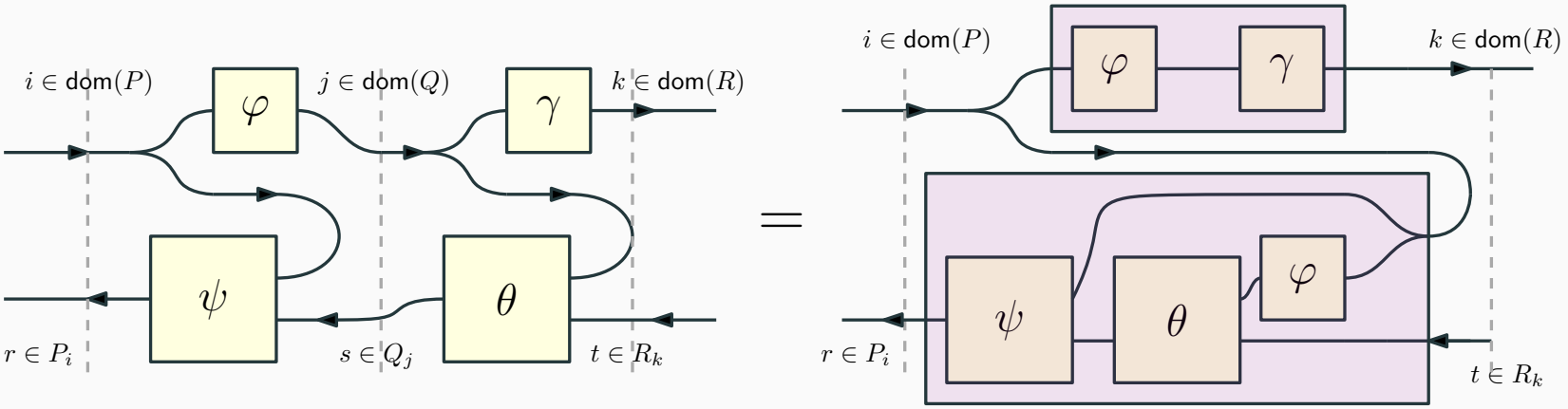
Official definition

$P \leq_{\mathrm{W}} Q$ if there are **computable**

$$f : \mathrm{dom}(P) \to \mathrm{dom}(Q) \quad \text{and} \quad F : \prod_{i \in \mathrm{dom}(P)} (Q_{f(i)} \to P_i)$$

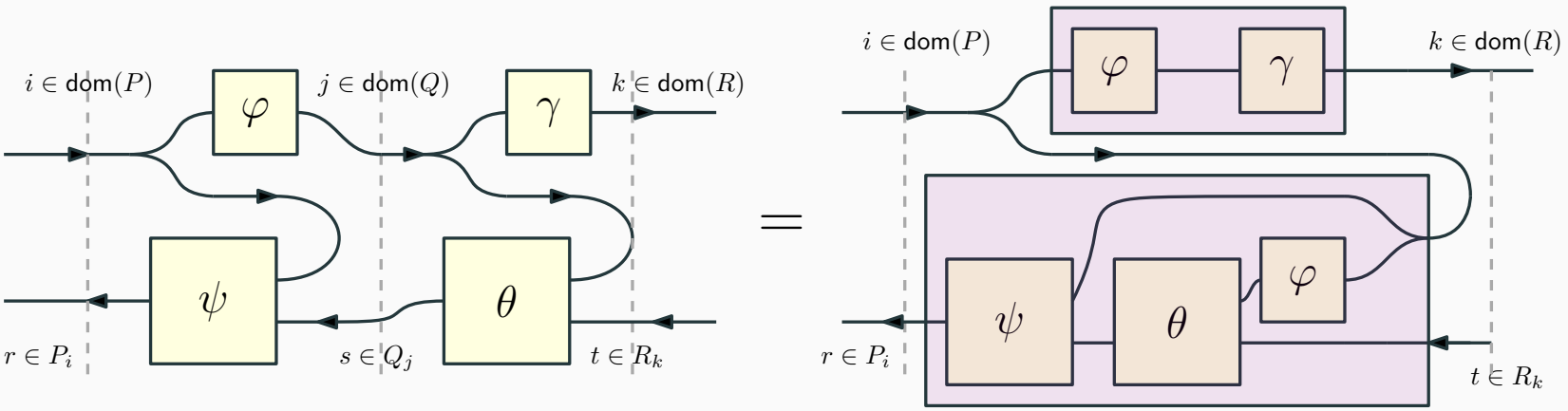


Sanity checks



Reductions compose + Quotienting by $\equiv_{\mathrm{W}} \rightsquigarrow$ **Weihrauch degrees**

Sanity checks



Reductions compose + Quotienting by $\equiv_{\mathrm{W}} \rightsquigarrow$ **Weihrauch degrees**

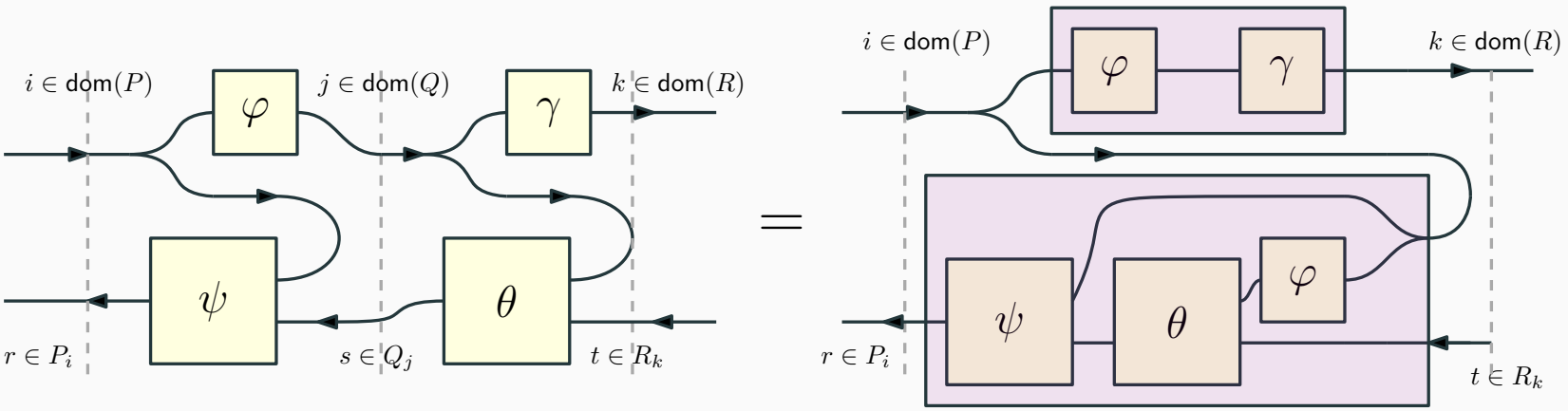
Abstractly: reductions are container morphisms

Weihrauch problems = containers over represented subspaces of $2^{\mathbb{N}}$

Makes sense in over any category with pullbacks

($\simeq$ internal families make sense)

Sanity checks



Reductions compose + Quotienting by $\equiv_{\mathrm{W}} \rightsquigarrow$ **Weihrauch degrees**

Abstractly: reductions are container morphisms

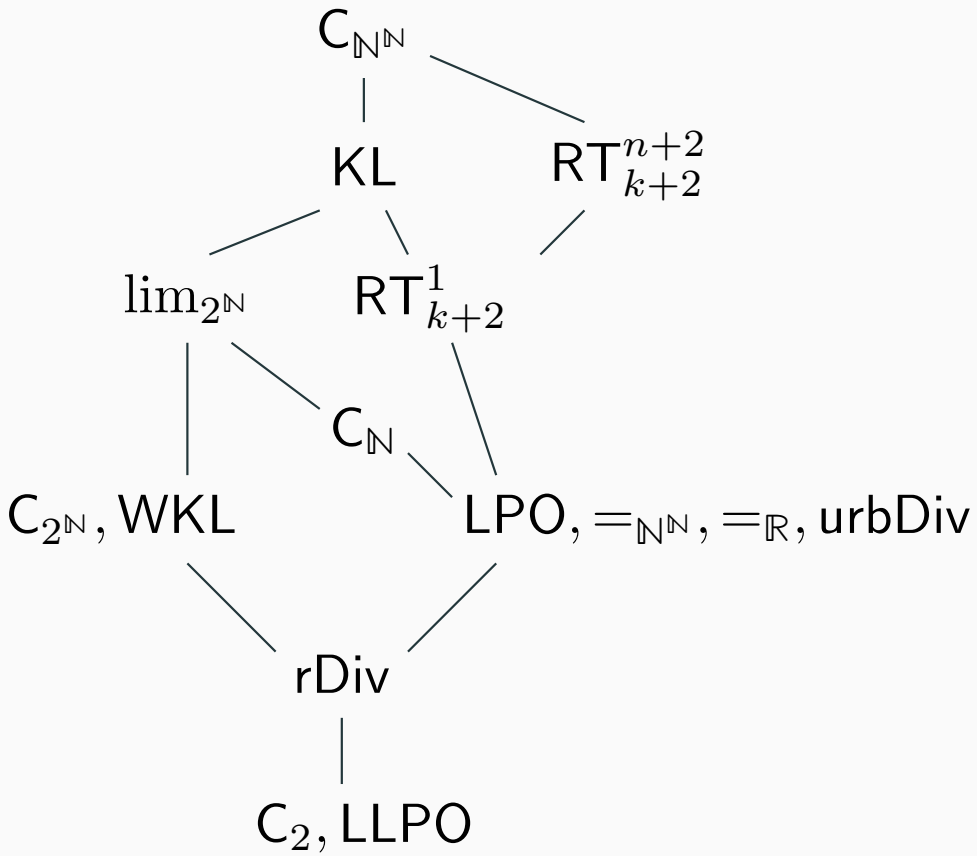
Weihrauch problems = containers over represented subspaces of $2^{\mathbb{N}}$

Makes sense in over any category with pullbacks

($\simeq$ internal families make sense)

For LCCCs: "reductions" $\cong$ strong natural transformations between polynomial functors $X \longmapsto \sum_{i \in \mathrm{dom}(P)} X^{P_i}$

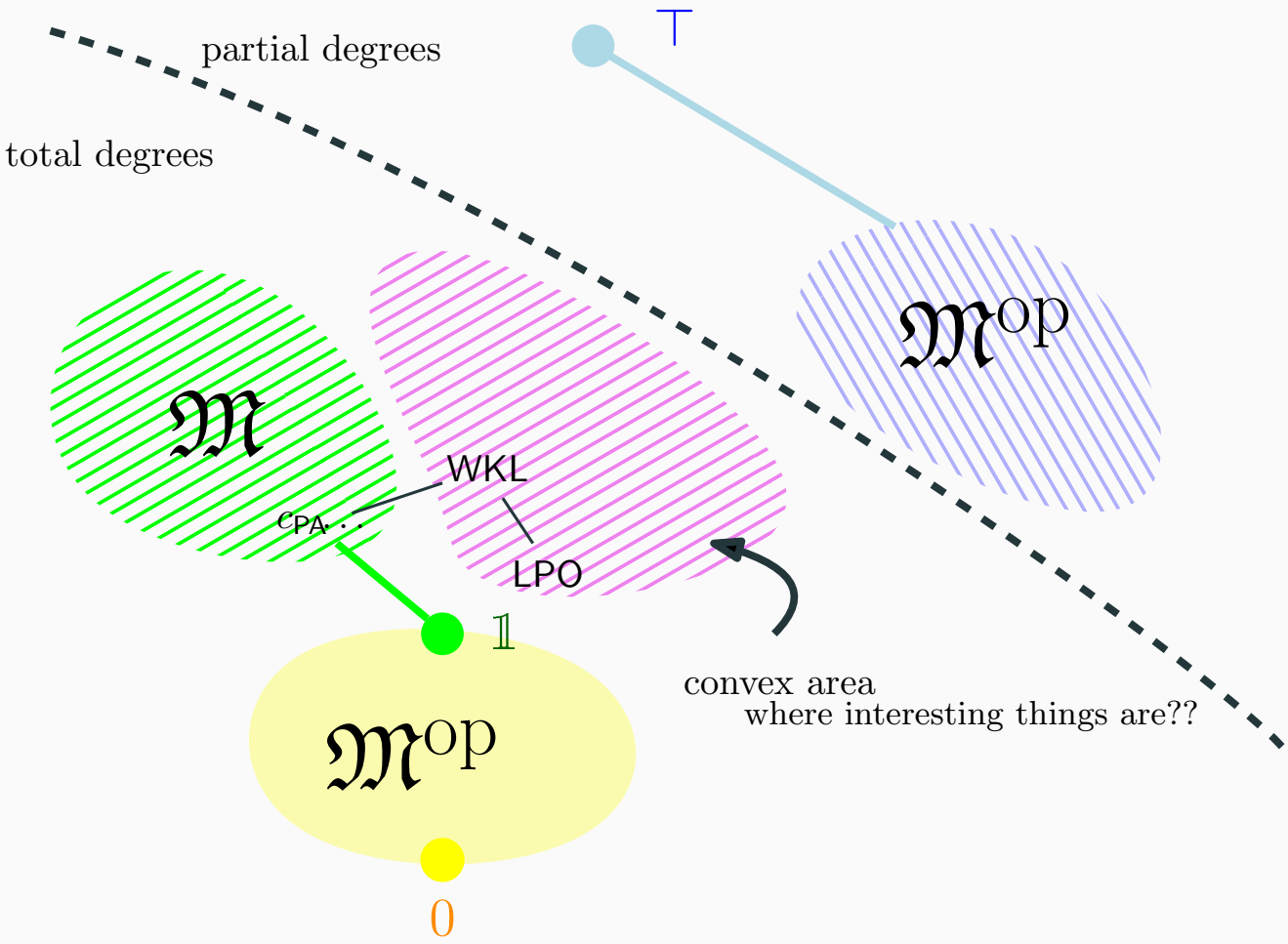
How some problems we've seen so far relate



One can certainly draw more ambitious zoos...

Zooming out

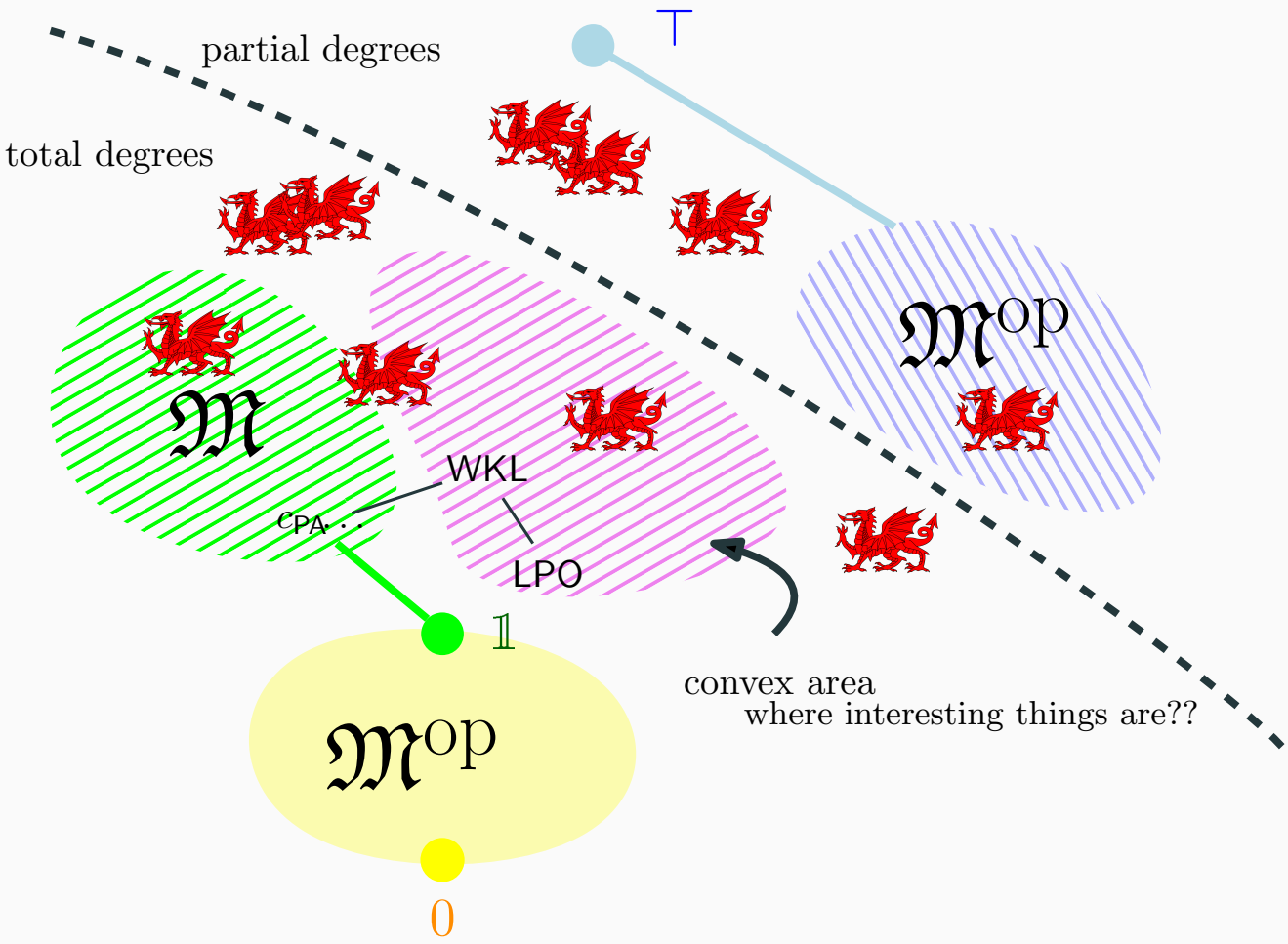
The (partial) Weihrauch lattice is big: $2^{2^{\aleph_0}}$.



$\mathfrak{M}$: the Medvedev lattice (also $2^{2^{\aleph_0}}$–big) (∼ only forward reductions)

Zooming out

The (partial) Weihrauch lattice is big: $2^{2^{\aleph_0}}$.



$\mathfrak{M}$: the Medvedev lattice (also $2^{2^{\aleph_0}}$–big) (∼ only forward reductions)

Technical asides: encoding and intensionality

Legitimate question: why subsets of $\mathbb{N}^{\mathbb{N}}$?

Why not take internal families/containers in *represented spaces*?

- It would make sense and the category is nicer
- But computability theorists want reductions to be **intensional**, which yields more identifications than working extensional.

$=_{\mathbb{S}}$, LPO, $=_{\mathbb{R}}$ intensionally equivalent, but not extensionally

Technical asides: encoding and intensionality

Legitimate question: why subsets of $\mathbb{N}^{\mathbb{N}}$?

Why not take internal families/containers in *represented spaces*?

- It would make sense and the category is nicer
- But computability theorists want reductions to be **intensional**, which yields more identifications than working extensional.

$=_{\mathbb{S}}$, LPO, $=_{\mathbb{R}}$ intensionally equivalent, but not extensionally

Tradeoffs

- Containers over a **weakly** locally cartesian closed category
- "Wlog": space $\rightsquigarrow$ the domain of its representation

Technical asides: encoding and intensionality

Legitimate question: why subsets of $\mathbb{N}^{\mathbb{N}}$?

Why not take internal families/containers in *represented spaces*?

- It would make sense and the category is nicer
- But computability theorists want reductions to be **intensional**, which yields more identifications than working extensional.

$=_{\mathbb{S}}$, LPO, $=_{\mathbb{R}}$ intensionally equivalent, but not extensionally

Tradeoffs

- Containers over a **weakly** locally cartesian closed category
- "Wlog": space $\rightsquigarrow$ the domain of its representation

Rough take-away: a lot of **algebra** from the theory of polynomial functors works, **up to annoying details**

Technical asides: encoding and intensionality

Legitimate question: why subsets of $\mathbb{N}^{\mathbb{N}}$?

Why not take internal families/containers in *represented spaces*?

- It would make sense and the category is nicer
- But computability theorists want reductions to be **intensional**, which yields more identifications than working extensional.

$=_{\mathbb{S}}$, LPO, $=_{\mathbb{R}}$ intensionally equivalent, but not extensionally

Tradeoffs

- Containers over a **weakly** locally cartesian closed category
- "Wlog": space $\rightsquigarrow$ the domain of its representation

Rough take-away: a lot of **algebra** from the theory of polynomial functors works, **up to annoying details**

(or developping the theory of polynomial quasifunctors if you're volunteering)

Algebras of problems

Some algebra on problems

A lot of generic nice structure.

- Joins $+$: "solve one of the two problems"

$$\mathrm{dom}(P+Q) \cong \mathrm{dom}(P) + \mathrm{dom}(Q) \qquad \begin{array}{rcl} (P+Q)_{\mathsf{in}_1(i)} & = & P_i \\ (P+Q)_{\mathsf{in}_2(j)} & = & Q_j \end{array}$$

- Meets $\wedge$: "given inputs for both, solve one"

$$\mathrm{dom}(P \wedge Q) \cong \mathrm{dom}(P) \times \mathrm{dom}(Q) \quad (P \wedge Q)_{i,j} = P_i + Q_j$$

- Parallel product $\otimes$: "solve both problems"

$$\mathrm{dom}(P \otimes Q) \cong \mathrm{dom}(P) \times \mathrm{dom}(Q) \quad (P \otimes Q)_{i,j} = P_i \times Q_j$$

Exercise: what are the units? What are the corresponding operations on polynomials?

Some examples

- Most relationships with $+, \wedge$

  $\rightsquigarrow$ deduce from the theory of distributive lattices.
- $\mathsf{LPO} <_{\mathrm{W}} \mathsf{LPO} \otimes \mathsf{LPO} <_{\mathrm{W}} \ldots <_{\mathrm{W}} \mathsf{LPO}^n$
- $\mathsf{LPO}^n <_{\mathrm{W}} \mathsf{RT}^2_2$
- Some problems P are **finitely parallelizable**:

$$P \otimes P \leq_{\mathrm{W}} P \qquad \text{and} \qquad \mathbb{1} \leq_{\mathrm{W}} P$$

Examples: WKL_0, $\lim_{2^{\mathbb{N}}}$, $\mathsf{C}_{\mathbb{N}}$, KL, $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

Some examples

- Most relationships with $+, \wedge$

  $\rightsquigarrow$ deduce from the theory of distributive lattices.
- $\mathsf{LPO} <_{\mathrm{W}} \mathsf{LPO} \otimes \mathsf{LPO} <_{\mathrm{W}} \ldots <_{\mathrm{W}} \mathsf{LPO}^n$
- $\mathsf{LPO}^n <_{\mathrm{W}} \mathsf{RT}^2_2$
- Some problems P are **finitely parallelizable**:

$$P \otimes P \leq_{\mathrm{W}} P \qquad \text{and} \qquad \mathbb{1} \leq_{\mathrm{W}} P$$

Examples: WKL_0, $\lim_{2^{\mathbb{N}}}$, $\mathsf{C}_{\mathbb{N}}$, KL, $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

Definition (?)

The **finite parallelization** of P is given by

$$P^* \quad = \quad \mu X.\, \mathbb{1} + P \otimes X$$

Some examples

- Most relationships with $+, \wedge$

  $\rightsquigarrow$ deduce from the theory of distributive lattices.
- $\mathsf{LPO} <_{\mathrm{W}} \mathsf{LPO} \otimes \mathsf{LPO} <_{\mathrm{W}} \ldots <_{\mathrm{W}} \mathsf{LPO}^n$
- $\mathsf{LPO}^n <_{\mathrm{W}} \mathsf{RT}^2_2$
- Some problems P are **finitely parallelizable**:

$$P \otimes P \leq_{\mathrm{W}} P \qquad \text{and} \qquad \mathbb{1} \leq_{\mathrm{W}} P$$

Examples: WKL_0, $\lim_{2^{\mathbb{N}}}$, $\mathsf{C}_{\mathbb{N}}$, KL, $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

Definition

The **finite parallelization** of P is given by

$$P^* \quad = \quad \mu X.\, \mathbb{1} + P \otimes X$$

$$\mathrm{dom}(P^*) = \mathrm{dom}(P)^* \qquad\qquad P_w = \prod_{i \in |w|} P_{w_i}$$

One oracle call after the other

Sequential composition $P \star Q$

- Implicitly: ability to make an oracle call to Q then P
- Explicitly: given an instance i of Q and a function that takes a solution of i to an instance of P, compute all relevant solutions

$$\begin{aligned} \mathrm{dom}(P \star Q) &= \sum_{i \in \mathrm{dom}(Q)} \left(Q_i \xrightarrow{\sim} \mathrm{dom}(P)\right) \\ (P \star Q)_{i,f} &= \sum_{r \in Q_i} P_{f(r)} \qquad \text{(using the weak exponential } \xrightarrow{\sim} \dots\text{)} \end{aligned}$$

One oracle call after the other

Sequential composition $P \star Q$

- Implicitly: ability to make an oracle call to Q then P
- Explicitly: given an instance i of Q and a function that takes a solution of i to an instance of P, compute all relevant solutions

$$\begin{aligned} \mathrm{dom}(P \star Q) &= \sum_{i \in \mathrm{dom}(Q)} \left(Q_i \xrightarrow{\sim} \mathrm{dom}(P)\right) \\ (P \star Q)_{i,f} &= \sum_{r \in Q_i} P_{f(r)} \qquad \text{(using the weak exponential } \xrightarrow{\sim} \dots\text{)} \end{aligned}$$

Extensional version, on polynomials

$$\left(\sum_{i \in \mathrm{dom}(P)} X^{P_i}\right) \circ \left(\sum_{j \in \mathrm{dom}(Q)} X^{Q_j}\right) \cong \sum_{\substack{j \in \mathrm{dom}(Q) \\ f : Q_j \to \mathrm{dom}(P)}} X^{\sum_r P_{f(r)}}$$

Some examples

- Obviously, $P \otimes Q \leq_{\mathrm{W}} P \star Q$.
- $\mathsf{LPO} \otimes \mathsf{LPO} <_{\mathrm{W}} \mathsf{LPO} \star \mathsf{LPO} <_{\mathrm{W}} \mathsf{LPO} \otimes \mathsf{LPO} \otimes \mathsf{LPO}$
- Some problems P are closed under sequential composition

$$P \star P \leq P \qquad\qquad 1 \leq P$$

Examples: $\mathsf{C}_{\mathbb{N}}, \mathsf{WKL}_0, \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

Iterated composition $P^\diamond$

- Explicitly: computed as the least fixpoint of $X \mapsto \mathbb{1} + X \star P$
- Implicitly: ability to make a finite but not fixed in advance number of oracle calls to P

Weihrauch reducbility and Kleene algebras?

Equational theory of the Weihrauch lattice

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, +, \wedge)$

  (via the Medvedev degrees)
- Thus, $(\mathfrak{W}, +, \wedge) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

Equational theory of the Weihrauch lattice

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, +, \wedge)$

  (via the Medvedev degrees)
- Thus, $(\mathfrak{W}, +, \wedge) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

Meta-question

For a given signature, is there anything true in the Weihrauch degree that is not true for **all** (suitable) categories of containers?

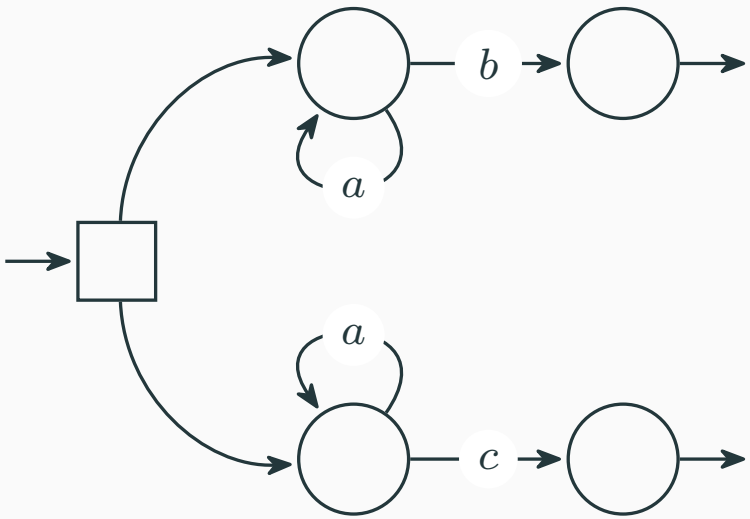
Terms with composition and automata

Starting observation

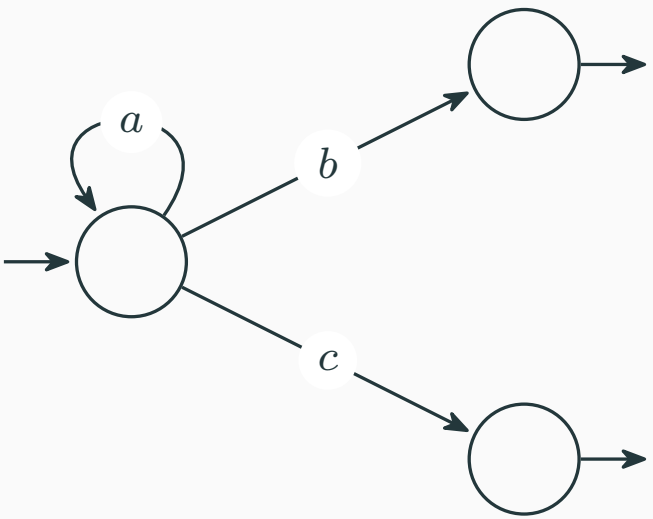
Terms over $0, \mathbb{1}, +, \star, (-)^\diamond, \top$ = almost regular expressions.

(alphabet = the set of variables)

- Terms can be mapped to NFAs in a meaningful way
- Adding $\wedge$ = allowing alternating automata



$b \star a^\diamond \ \wedge \ c \star a^\diamond$



$(b + c) \star a^\diamond$

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\eth(\mathcal{A}, \mathcal{B})$ that captures a variant of the notion of **simulation** such that

Theorem

$(\mathfrak{W}, \mathbb{1}, +, \top, \wedge, \star, (-)^\diamond) \models t \leq u$ iff Duplicator wins in $\eth(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\eth(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel

(using $\mathcal{B}$ exactly once)

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\eth(\mathcal{A}, \mathcal{B})$ that captures a variant of the notion of **simulation** such that

Theorem

$(\mathfrak{W}, \mathbb{1}, +, \top, \wedge, \star, (-)^\diamond) \models t \leq u$ iff Duplicator wins in $\eth(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\eth(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel

(using $\mathcal{B}$ exactly once)

$\Rightarrow$ $\mathcal{O}\left(|\mathcal{B}|2^{|\mathcal{A}|}\right)$ positions

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\eth(\mathcal{A}, \mathcal{B})$ that captures a variant of the notion of **simulation** such that

Theorem

$(\mathfrak{W}, \mathbb{1}, +, \top, \wedge, \star, (-)^\diamond) \models t \leq u$ iff Duplicator wins in $\eth(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\eth(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel

(using $\mathcal{B}$ exactly once)

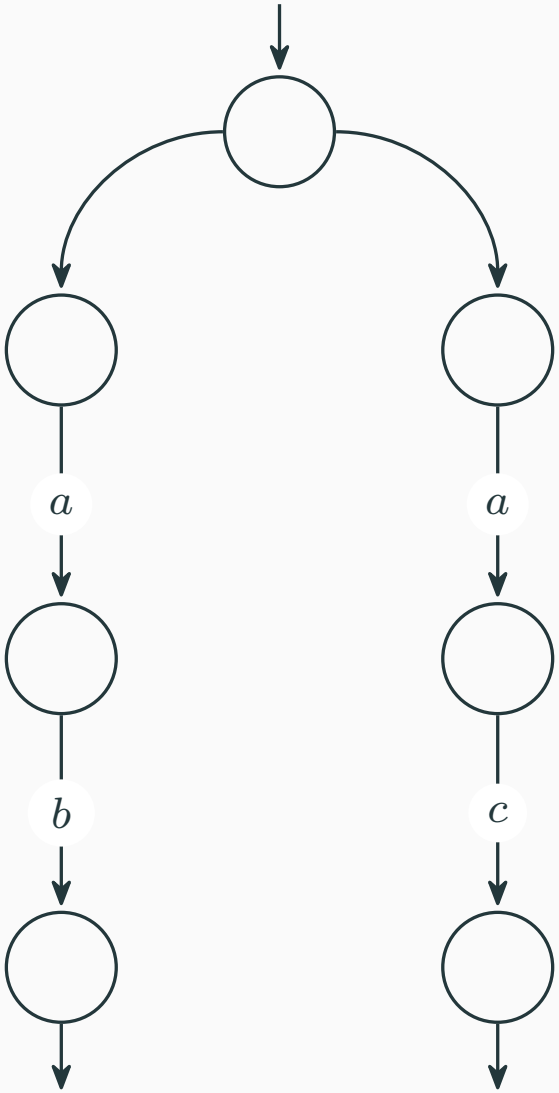
$\Rightarrow$ $\mathcal{O}\left(|\mathcal{B}|2^{|\mathcal{A}|}\right)$ positions

Corollary

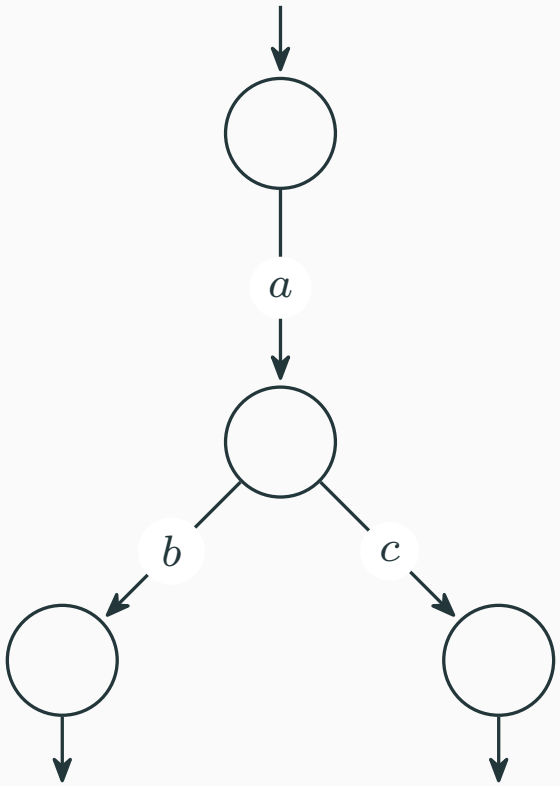
"$(\mathfrak{W}, \mathbb{1}, 0, \top, +, \wedge, \star, (-)^\diamond) \models t \leq u$?" is decidable.

- Conjecture: this is PSPACE-complete.

A simple example of simulation and non-simulation

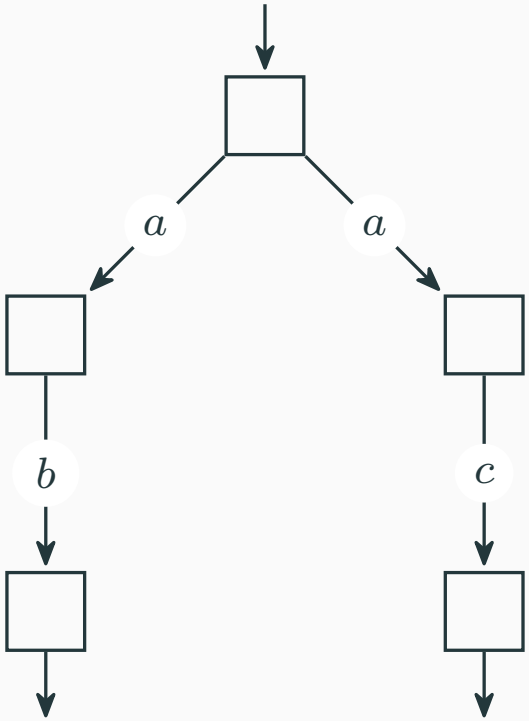


$(b \star a) + (c \star a)$ $<$

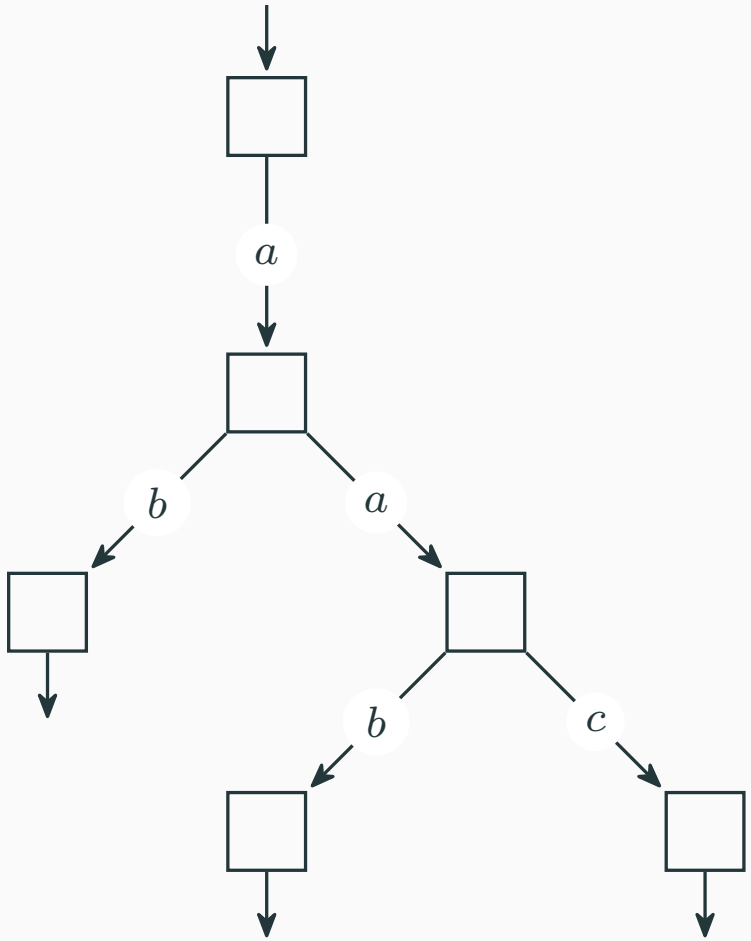


$(b + c) \star a$

A simulation requiring several concurrent attempts

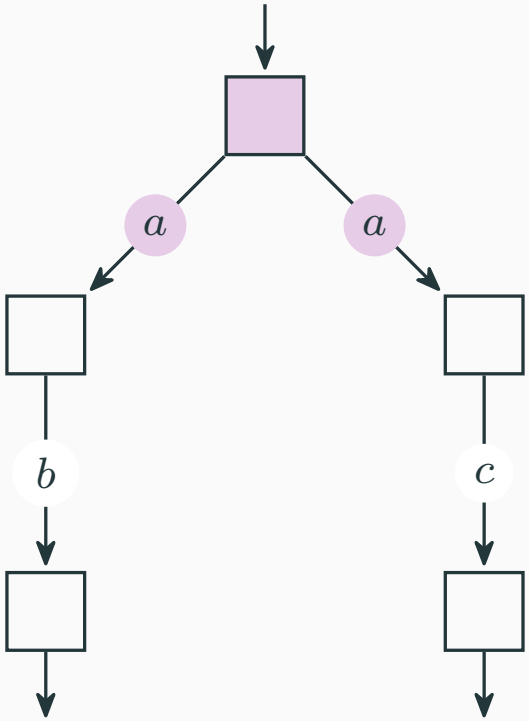


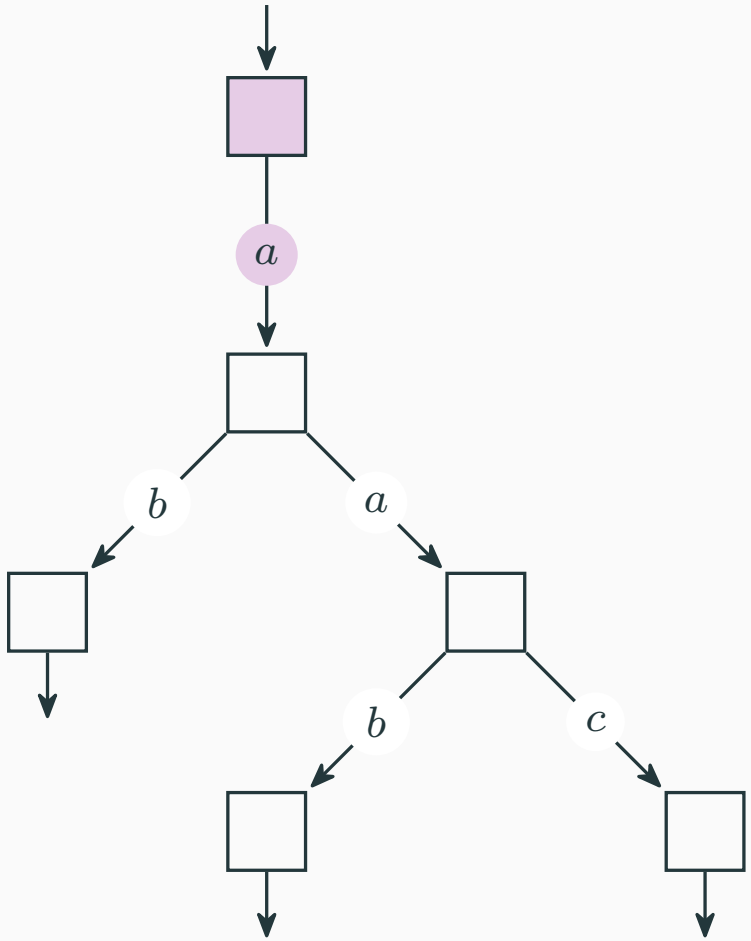
$(b \star a) \wedge (c \star a)$ $<$



$(((c \wedge b) \star a \ \wedge \ b) \star a$

A simulation requiring several concurrent attempts

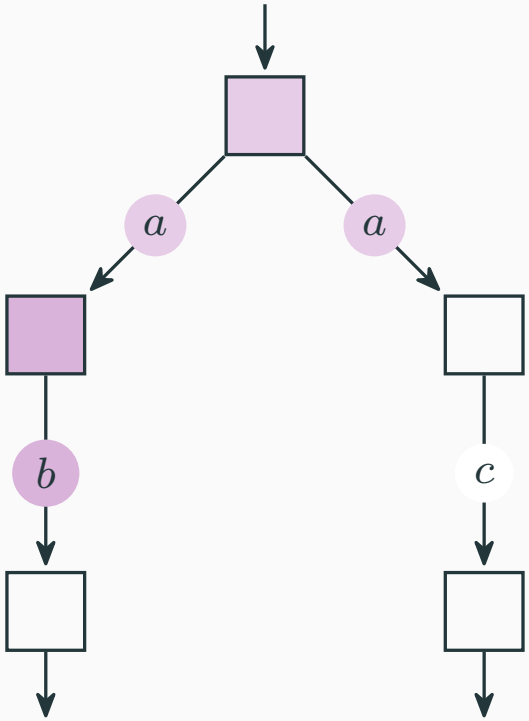


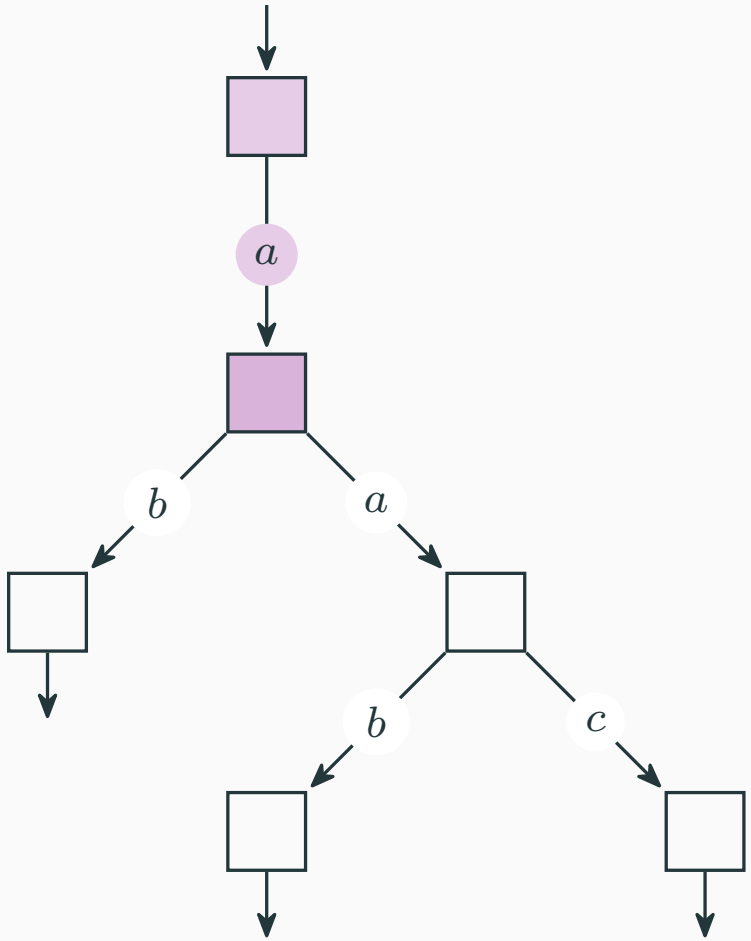


$(b \star a) \wedge (c \star a) \quad < \quad (((c \wedge b) \star a \ \wedge \ b) \star a$

Let's play ?!

A simulation requiring several concurrent attempts

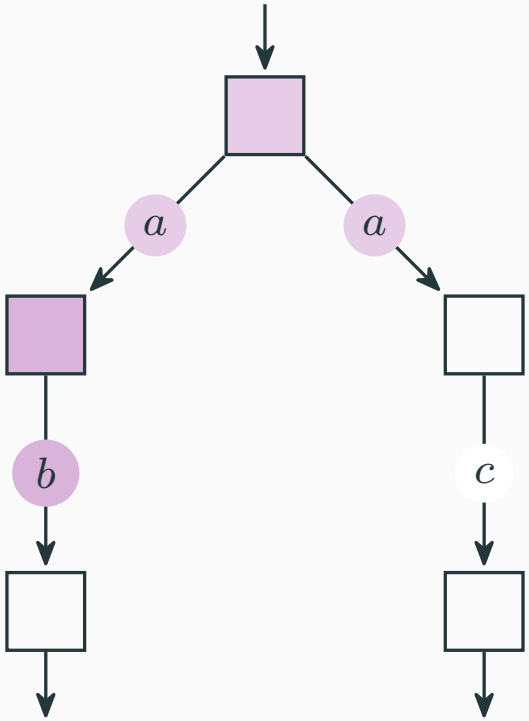


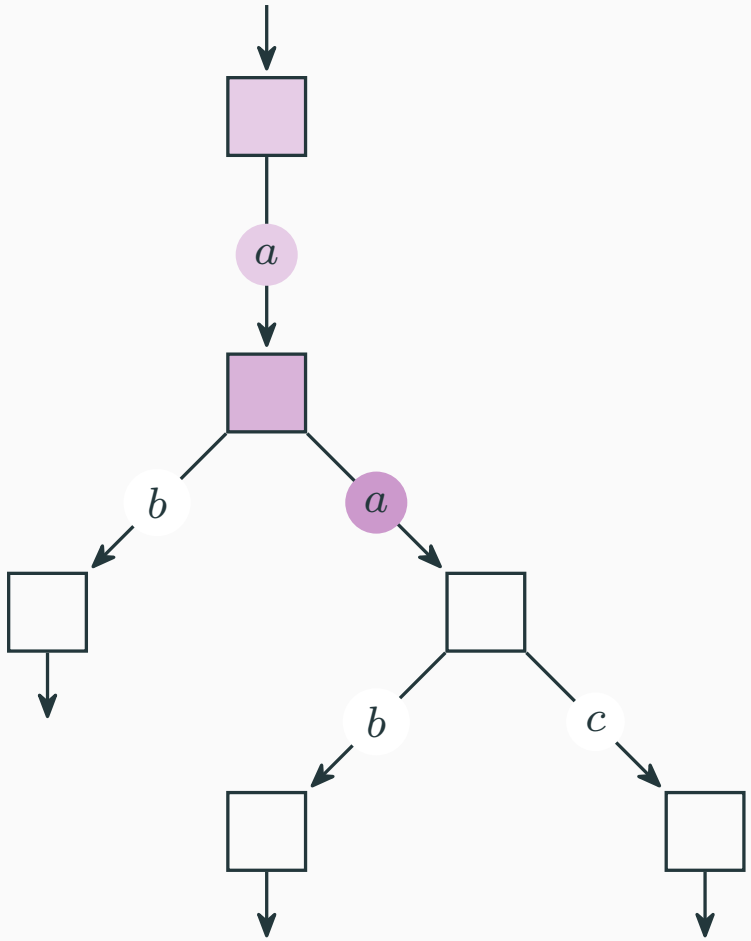


$(b \star a) \wedge (c \star a) \quad < \quad (((c \wedge b) \star a \ \wedge \ b) \star a$

Let's play ?!

A simulation requiring several concurrent attempts

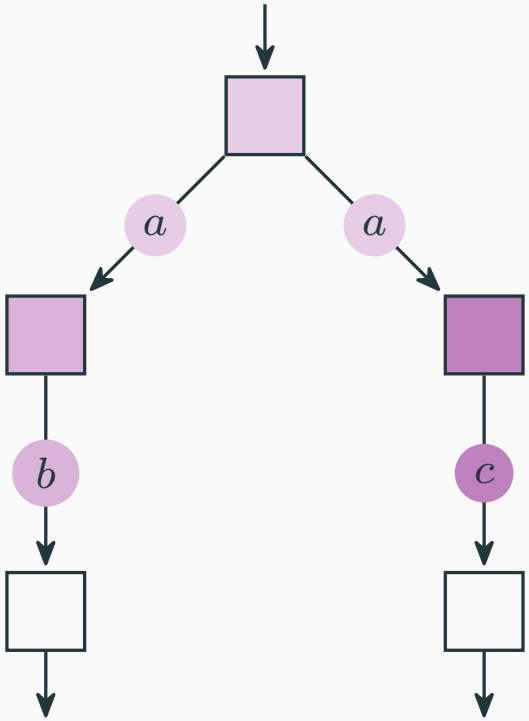


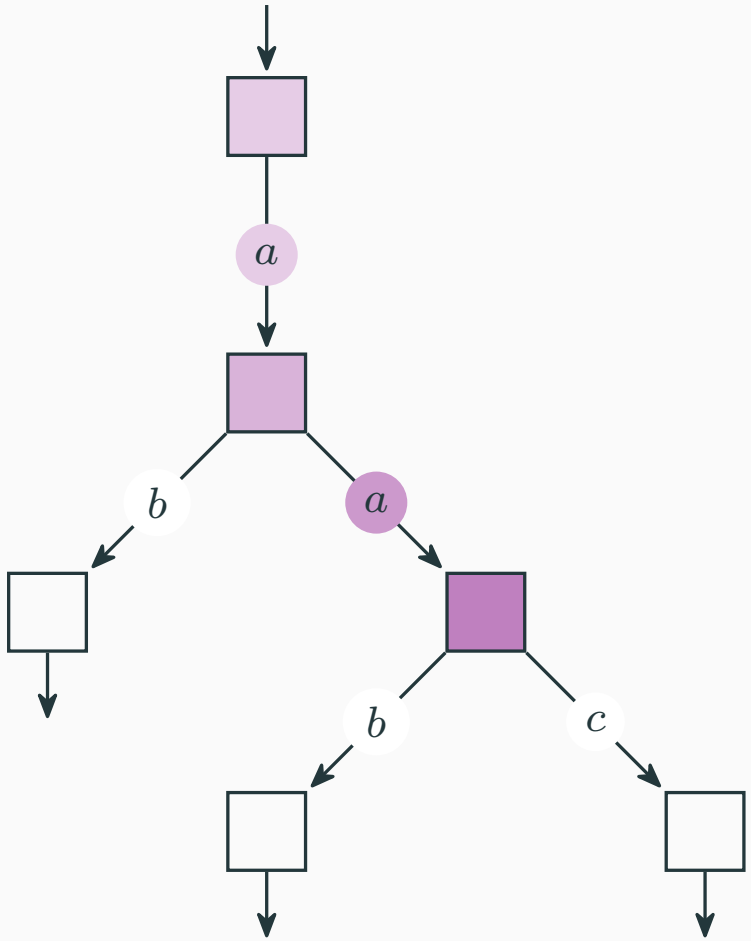


$(b \star a) \wedge (c \star a) \quad < \quad (((c \wedge b) \star a \ \wedge \ b) \star a$

Let's play ?!

A simulation requiring several concurrent attempts

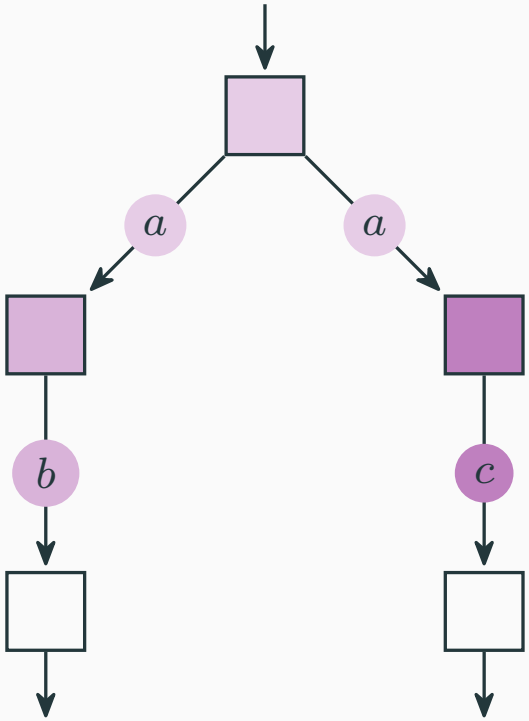


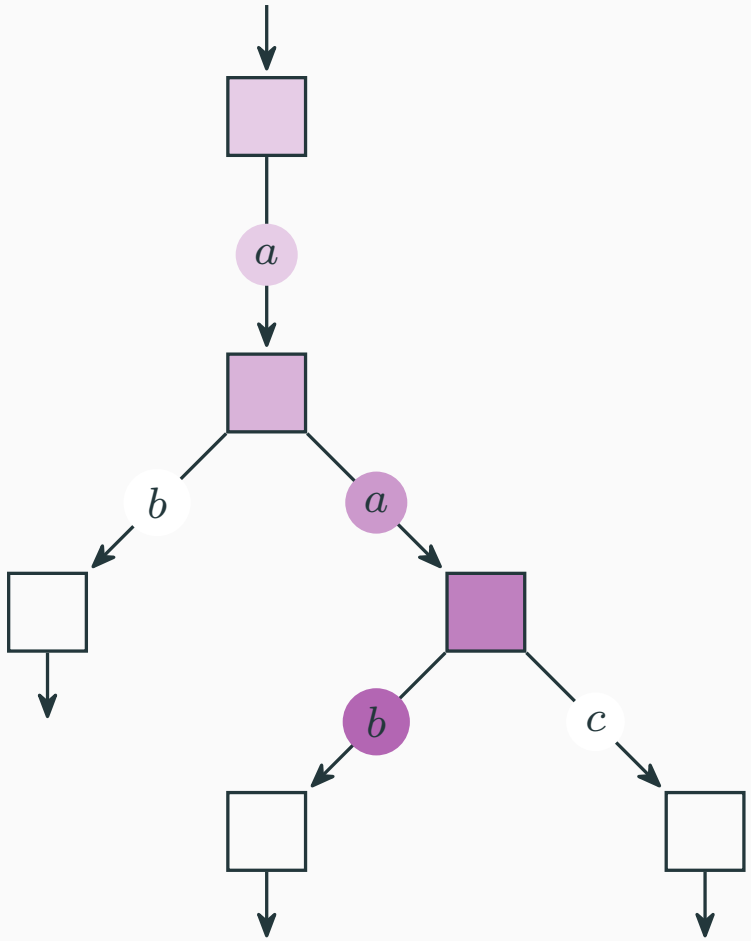


$(b \star a) \wedge (c \star a) \quad < \quad (((c \wedge b) \star a \ \wedge \ b) \star a$

Let's play ?!

A simulation requiring several concurrent attempts

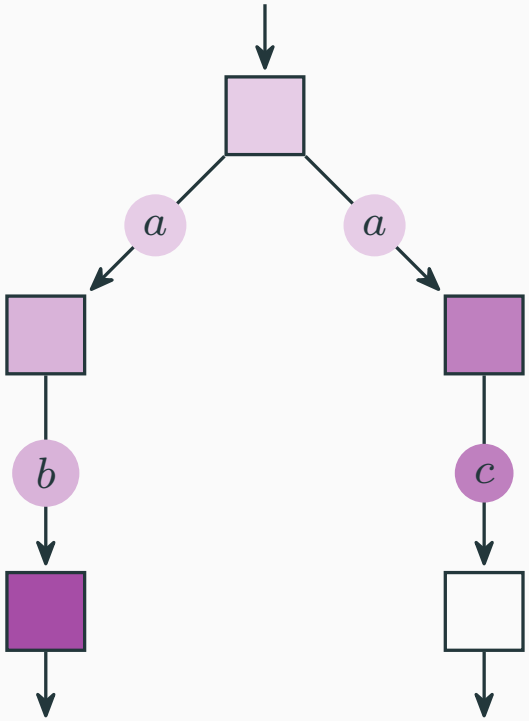


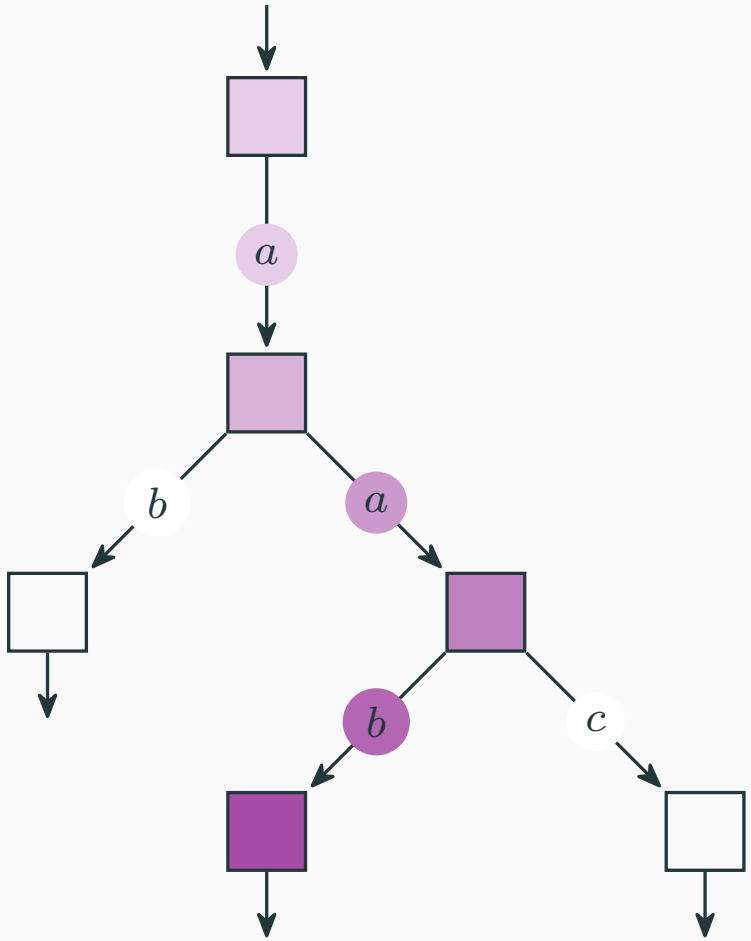


$(b \star a) \wedge (c \star a) \quad < \quad (((c \wedge b) \star a \ \wedge \ b) \star a$

Let's play ?!

A simulation requiring several concurrent attempts

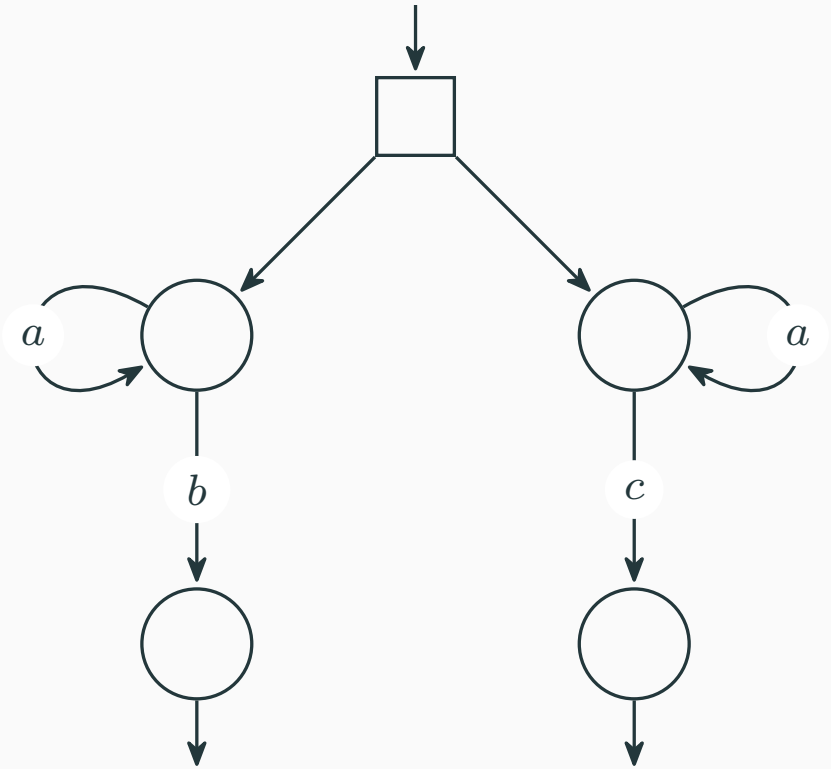




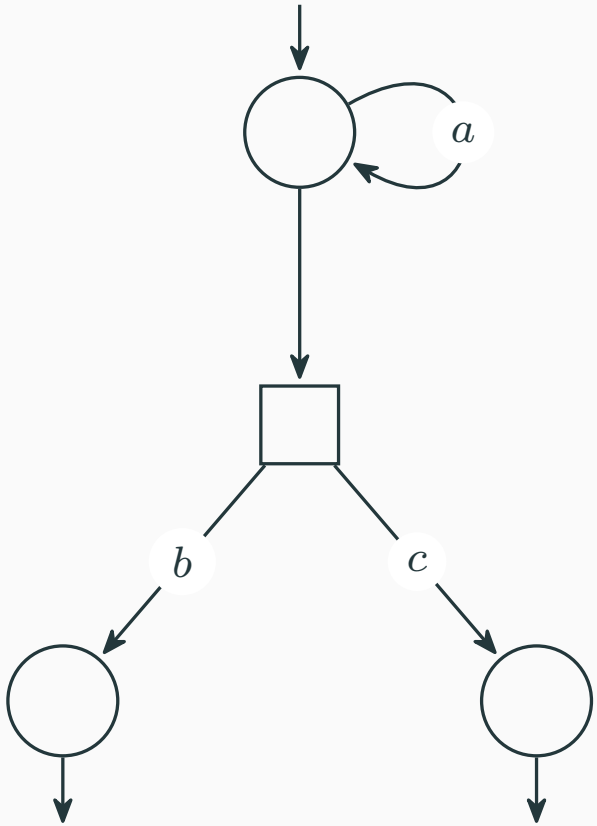
$(b \star a) \wedge (c \star a)$ $<$ $(((c \wedge b) \star a \ \wedge \ b) \star a$

Let's play ?!

Another simulation requiring several concurrent attempts



$(b \star a^{\diamond}) \wedge (c \star a^{\diamond})$ $\equiv$



$(b \wedge c) \star a^{\diamond}$

Where do the concurrent attempt come from

- $\mathrm{dom}(a \wedge b) = \mathrm{dom}(a) \times \mathrm{dom}(b)$,
- Solving either question is progress along a possible run
- Conceptually captured in the following **strength** axiom:

$$a \star b \ \wedge \ c \quad \leq_{\mathrm{W}} \quad (a \wedge c) \star b$$

Where do the concurrent attempt come from

- $\mathrm{dom}(a \wedge b) = \mathrm{dom}(a) \times \mathrm{dom}(b)$,
- Solving either question is progress along a possible run
- Conceptually captured in the following **strength** axiom:

$$a \star b \ \wedge \ c \quad \leq_{\mathrm{W}} \quad (a \wedge c) \star b$$

For category theorists

Corresponds to the canonical strength in polynomial functors.

$$P(X) \times Y \quad \longleftarrow \quad P(X \times Y)$$

Induction principles for $(-)^\diamond$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \star x \leq x \qquad \Rightarrow \qquad x^\diamond \leq x$$

Induction principles for $(-)^\diamond$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \star x \leq x \qquad \Rightarrow \qquad x^\diamond \leq x$$

- Similar: an axiom of **left-handed Kleene algebras** (LKA)

$$x \star y \leq x \qquad \Rightarrow \qquad x \star y^\diamond \leq x$$

Induction principles for $(-)^\diamond$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \star x \leq x \qquad \Rightarrow \qquad x^\diamond \leq x$$

- Similar: an axiom of **left-handed Kleene algebras** (LKA)

$$x \star y \leq x \qquad \Rightarrow \qquad x \star y^\diamond \leq x$$

- For $\wedge$, we additionally need

$$x \star y \ \wedge \ z \leq x \qquad \Rightarrow \qquad x \star y^\diamond \ \wedge \ z \leq x$$

(exercise: $\mathbb{1} \leq a \wedge b$ implies $a^\diamond \wedge b^\diamond \leq (a \wedge b)^\diamond$)

Induction principles for $(-)^\diamond$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \star x \leq x \qquad \Rightarrow \qquad x^\diamond \leq x$$

- Similar: an axiom of **left-handed Kleene algebras** (LKA)

$$x \star y \leq x \qquad \Rightarrow \qquad x \star y^\diamond \leq x$$

- For $\wedge$, we additionally need

$$x \star y \ \wedge \ z \leq x \qquad \Rightarrow \qquad x \star y^\diamond \ \wedge \ z \leq x$$

(exercise: $\mathbb{1} \leq a \wedge b$ implies $a^\diamond \wedge b^\diamond \leq (a \wedge b)^\diamond$)

Theorem

The above axioms are valid in the partial Weihrauch degrees.

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **w/o right-distributivity of $+$ over $\star$**

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **w/o right-distributivity of $+$ over $\star$**
  - i.e. it can be the case that $(P + Q) \star R \not\equiv_{\mathrm{W}} (P \star R) + (Q \star R)$

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **w/o right-distributivity of $+$ over $\star$**
  - i.e. it can be the case that $(P + Q) \star R \not\equiv_{\mathrm{W}} (P \star R) + (Q \star R)$
  - why: LKA $=$ language inclusions, but we want simulations

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **w/o right-distributivity of $+$ over $\star$**
  - i.e. it can be the case that $(P + Q) \star R \not\equiv_{\mathrm{W}} (P \star R) + (Q \star R)$
  - why: LKA $=$ language inclusions, but we want simulations
- The distributive lattice axioms **with units** $+$

$$
\begin{array}{rcll}
x \star y \ \wedge \ x \star z & \leq & x \star (y \wedge z) & \text{distributivity} \\
(x \star y) \wedge z & \leq & (x \wedge z) \star y & \text{strength} \\
(x \star y) \wedge z \leq x & \Rightarrow & (x \star y^{\diamond}) \wedge z \leq x & \\
x \star \top = \top & & x \star 0 = 0 &
\end{array}
$$

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **w/o right-distributivity of $+$ over $\star$**
  - i.e. it can be the case that $(P + Q) \star R \not\equiv_{\mathrm{W}} (P \star R) + (Q \star R)$
  - why: LKA $=$ language inclusions, but we want simulations
- The distributive lattice axioms **with units** $+$

$$
\begin{array}{rcll}
x \star y \ \wedge \ x \star z & \leq & x \star (y \wedge z) & \text{distributivity} \\
(x \star y) \wedge z & \leq & (x \wedge z) \star y & \text{strength} \\
(x \star y) \wedge z \leq x & \Rightarrow & (x \star y^{\diamond}) \wedge z \leq x & \\
x \star \top = \top & & x \star 0 = 0 &
\end{array}
$$

Theorem

Complete for the equational theory of $(\mathfrak{W}, \mathbb{1}, 0, \top, +, \wedge, \star, (-)^{\diamond})$.

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **w/o right-distributivity of $+$ over $\star$**
  - i.e. it can be the case that $(P + Q) \star R \not\equiv_{\mathrm{W}} (P \star R) + (Q \star R)$
  - why: LKA $=$ language inclusions, but we want simulations
- The distributive lattice axioms **with units** $+$

$$
\begin{array}{rcll}
x \star y \ \wedge \ x \star z & \leq & x \star (y \wedge z) & \text{distributivity} \\
(x \star y) \wedge z & \leq & (x \wedge z) \star y & \text{strength} \\
(x \star y) \wedge z \leq x & \Rightarrow & (x \star y^{\diamond}) \wedge z \leq x & \\
x \star \top = \top & & x \star 0 = 0 &
\end{array}
$$

Theorem

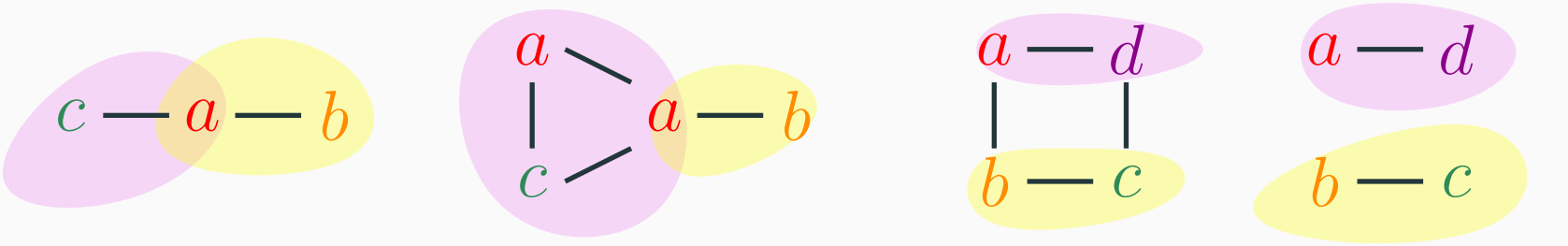
Complete for the equational theory of $(\mathfrak{W}, \mathbb{1}, 0, \top, +, \wedge, \star, (-)^{\diamond})$.

Proof idea: $\exists$ positional simulation strategies, induction on the syntax

How this started: the theory of $\otimes, \wedge$

A notion of combinatorial reduction between graphs

A reduction from (V_0, E_0, c_0) to (V_1, E_1, c_1) is a colour-preserving function $h : V_1 \to V_0$ such that the image of any maximal clique in (V_1, E_1) under h contains a maximal clique in (V_0, E_0).



$a \otimes (c \wedge b) \quad < \quad a \otimes ((a \otimes c) \wedge b) \qquad (a \wedge c) \otimes (b \wedge d) \quad < \quad (a \otimes d) \wedge (b \otimes c)$

Combinatorial characterization (Neumann, Pauly, P.)

$(\mathfrak{W}, \wedge, \otimes) \models t \leq u$ iff there is a reduction from G_t to G_u.
As a result, deciding $(\mathfrak{W}, \wedge, \otimes) \models t \leq u$ is Σ_2^p-complete.

- Axiomatizing: harder!
- $+$, $\star$ and $\otimes$: opens the gates of hell (**concurrency theory**)

Concluding thoughts on this stuff

I feel ignorant (help?)

- Is this relevant/known to non-computability theorists?
- Do people care about the algebra of strong monotone operators on complete Heyting algebras?
- Literature on simulation between alternating automata?

Avenues for further work

- Horn theory?
- Fun extensions to the signature:
  - Easyish: μ-calculus like-formalism with $\star, \wedge, \vee$
  - Harder: that + handling $(-)^\omega$ or ζ **(long games)**
  - Not sure: this or that + handling $\otimes$ **(concurrency)**

Some connections to logic

Weihrauch reducibility and reverse maths

How reverse maths compares problems

$P \leq_{\text{revMaths}} Q$ iff a weak theory proves

$$(\forall j \in \mathrm{dom}(Q).\ \exists b \in Q_j) \quad \Longrightarrow \quad (\forall i \in \mathrm{dom}(P).\ \exists a \in P_i)$$

Often Weihrauch reducibility often refines reverse maths:

- "linear use" of oracles
- finitary problems like LPO don't trivialize

Not always:

- checking a reduction works can be done in the metatheory
  $\rightsquigarrow$ less metamathematics issues

Weihrauch reducibility and reverse maths

How reverse maths compares problems

$P \leq_{\text{revMaths}} Q$ iff a weak theory proves

$$(\forall j \in \mathrm{dom}(Q).\ \exists b \in Q_j) \quad \Longrightarrow \quad (\forall i \in \mathrm{dom}(P).\ \exists a \in P_i)$$

Often Weihrauch reducibility often refines reverse maths:

- "linear use" of oracles
- finitary problems like LPO don't trivialize

Not always:

- checking a reduction works can be done in the metatheory
  $\rightsquigarrow$ less metamathematics issues

Formalizing a system of linear arithmetic that matches Weihrauch reducibility: hard and weird, see [Uftring 21].

Modalities associated to problems

Definition

Given a problem P with $\mathrm{dom}(P) = I$, define

$$\begin{array}{rccc} \bigcirc_P : & \Omega & \longrightarrow & \Omega \\ & \varphi & \longmapsto & \exists i \in I.\, \forall o \in P_i.\, \varphi \end{array}$$

- Intuition for proving $\bigcirc_P \varphi$: if someone has an answer to a P-question of my choosing, I can prove φ.
- We always $\varphi \Rightarrow \bigcirc_P \varphi$ if I is inhabited.
- Only one call; $\bigcirc_P \bigcirc_P \varphi \not\Rightarrow \bigcirc_P \varphi$ in general
- number of other sanity checks can be made

$$\bigcirc_F \varphi \wedge (\forall i \in I. \exists o \in P_i) \Rightarrow \varphi \qquad \forall i \in I.\, \bigcirc_P (\exists o \in P_i) \qquad \ldots$$

Endofunctors associated to problems

Definition

Given a problem P with $\mathrm{dom}(P) = I$, define

$$\begin{array}{rccc} \bigcirc_F : & \mathsf{Set} & \longrightarrow & \mathsf{Set} \\ & X & \longmapsto & \{f : P_i \to X \mid f \text{ constant}, i \in I\}/_\sim \end{array}$$

- Having an $\tilde{x} \in \bigcirc_P X$: should you be able to solve an arbitrary P-challenge, you can get an $x \in X$!
- (any solution $\to$ same result)
- (identify things that ultimately yield the same $x \in X$)
- Modalities: functorial action on injections into 1.

An endofunctor in action

The problem $\mathsf{C}_{\omega+1,2}$

- **Input:** a decreasing sequence $s \in (\omega+1)^\omega$
- **Output:** $b \in 2$ equal to the parity of $\min(s)$ if $\min(s) \neq \omega$

Call η the canonical map $2^{\mathbb{N}} \to \bigcirc_{\mathsf{C}_{\omega+1,2}}(2^{\mathbb{N}})$

CBBB for $2^{\mathbb{N}}$ and continuous maps (Neumann, Pauly, P.)

In KV-realizability, for any injections $f, g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, there is a "bijection" $h : 2^{\mathbb{N}} \to \bigcirc_{\mathsf{C}_{\omega+1,2}}(2^{\mathbb{N}})$ such that, for every $p \in 2^{\mathbb{N}}$,

$$\bigcirc_{\mathsf{C}_{\omega+1,2}} \left(h(x) = \eta(f(x)) \quad \vee \quad h(x) = \eta(g^{-1}(x)) \right)$$

Extended Weihrauch reducibility

Modalities in Kleene-Vesley realizability [Kihara 21]

internally monotone $j : \Omega \to \Omega$ $\quad\cong\quad$ extended Weihrauch degrees

Extended Weihrauch problems P: [Bauer 22]

- a set $\mathrm{dom}(P) \subseteq 2^{\mathbb{N}}$ of questions
- a familiy of answer sets $P_i \subseteq \mathcal{P}(2^{\mathbb{N}})$, to be thought as secret/magical questions.

How do extended problems work?

- Arthur asks a question $i \in \mathrm{dom}(P)$
- Merlin asks a secret clarification $P_i \subseteq \mathcal{P}(2^{\mathbb{N}})$
- Nimue gives back some $a \in P_i$ to Arthur

Let's draw a picture on the board for reductions if there is time and interest.

What is the point?

- Fun new problems!
- Related to Lawvere-Tierney operators on realizability toposes
  - New independence proofs that are computability theoretic
  - Step towards classifying those [Kihara and collaborators]
- Even richer structure than Weihrauch degrees
  - Impredicative quantification
  - I.e., **propositions-as-problems** topos [Maschio & Trotta 25]

Suggested future work? (there'd be a lot)

- Modalities and fine-grained reverse mathematics?

  (if not too boring...)

  - Opportunity to go set-theoretic/higher-order?

For instance

Can we say something about "set divison" theorems?

$$X \times k \cong Y \times k \implies X \cong Y \qquad (k \in \mathbb{N})$$

Suggested future work? (there'd be a lot)

- Modalities and fine-grained reverse mathematics?

  (if not too boring...)

  - Opportunity to go set-theoretic/higher-order?

For instance

Can we say something about "set divison" theorems?

$$X \times k \cong Y \times k \implies X \cong Y \qquad (k \in \mathbb{N})$$

- Study the proposition-as-problems topos
  - Conjecture: it's boring for arithmetical statements
  - It's less boring for weird definability questions

$$\begin{aligned} \mathsf{RT}^1_2 &= \forall Z.\, (\forall P\, Q : \Omega.\, P^\infty \wedge Q^\infty \Rightarrow Z \star (P \wedge Q)^\infty) \Rightarrow Z \\ P \otimes Q &= \exists R\, S : \Omega.\, R \star P \wedge S \star Q \wedge (R \wedge S \Rightarrow \mathbb{1}) \end{aligned}$$

Suggested future work? (there'd be a lot)

- Modalities and fine-grained reverse mathematics?

  (if not too boring...)

  - Opportunity to go set-theoretic/higher-order?

For instance

Can we say something about "set divison" theorems?

$$X \times k \cong Y \times k \implies X \cong Y \qquad (k \in \mathbb{N})$$

- Study the proposition-as-problems topos
  - Conjecture: it's boring for arithmetical statements
  - It's less boring for weird definability questions

$$\begin{aligned} \mathsf{RT}^1_2 &= \forall Z.\, (\forall P\, Q : \Omega.\, P^\infty \wedge Q^\infty \Rightarrow Z \star (P \wedge Q)^\infty) \Rightarrow Z \\ P \otimes Q &= \exists R\, S : \Omega.\, R \star P \wedge S \star Q \wedge (R \wedge S \Rightarrow \mathbb{1}) \end{aligned}$$

Thanks for listening! Questions?

Other TCS links:
Fixpoints of problems

Motivation

We've seen some useful **least fixpoints**.

Other useful fixpoints

Infinite parallelization $\widehat{P}$: fixpoint of $X \mapsto P \otimes X$

$$\mathrm{dom}(\widehat{P}) = \mathrm{dom}(P)^{\mathbb{N}} \qquad \widehat{P}_s = \prod_{n \in \mathbb{N}} P_{s_n}$$

Infinite sequential iteration P^ω: fixpoint of $X \mapsto X \star P$

Construction as an inverse limit left to the listener

Motivation

We've seen some useful **least fixpoints**.

Other useful fixpoints

Infinite parallelization $\widehat{P}$: fixpoint of $X \mapsto P \otimes X$

$$\mathrm{dom}(\widehat{P}) \quad = \quad \mathrm{dom}(P)^{\mathbb{N}} \qquad \widehat{P}_s \quad = \quad \prod_{n \in \mathbb{N}} P_{s_n}$$

Infinite sequential iteration P^ω: fixpoint of $X \mapsto X \star P$

Construction as an inverse limit left to the listener

Exercise: these are **not** greatest fixpoints!

E.g. $\nu X.\, \mathsf{LPO} \otimes X \quad \equiv_{\mathrm{W}} \quad \nu X.\, X \star \mathsf{LPO} \quad \equiv_{\mathrm{W}}$

Motivation

We've seen some useful **least fixpoints**.

Other useful fixpoints

Infinite parallelization $\widehat{P}$: fixpoint of $X \mapsto P \otimes X$

$$\mathrm{dom}(\widehat{P}) \quad = \quad \mathrm{dom}(P)^{\mathbb{N}} \qquad \widehat{P}_s \quad = \quad \prod_{n \in \mathbb{N}} P_{s_n}$$

Infinite sequential iteration P^ω: fixpoint of $X \mapsto X \star P$

Construction as an inverse limit left to the listener

Exercise: these are **not** greatest fixpoints!

E.g. $\nu X.\, \mathsf{LPO} \otimes X \quad \equiv_{\mathrm{W}} \quad \nu X.\, X \star \mathsf{LPO} \quad \equiv_{\mathrm{W}} \quad \top$

Fixpoint of operators

Existence of useful fixpoints (P. & Price '26)

If F is a nice enough endofunctor over containers, over a nice enough category (whatever that means), the following exists:

- an initial algebra μF for F
- a terminal coalgebra νF for F
- a somewhat canonical (co)algebra ζF sitting in-between

- Idea: construct a fixpoint on domains, and then on the answers.
- Application: $\widehat{P} \equiv_{\mathrm{W}} \zeta X.\, X \otimes P$ and P^{ω} morally like $\zeta X.\, X \star P$.

Fixpoint of operators

Existence of useful fixpoints (P. & Price '26)

If F is a nice enough endofunctor over containers, over a nice enough category (whatever that means), the following exists:

- an initial algebra μF for F
- a terminal coalgebra νF for F
- a somewhat canonical (co)algebra ζF sitting in-between

- Idea: construct a fixpoint on domains, and then on the answers.
- Application: $\widehat{P} \equiv_{\mathrm{W}} \zeta X.\, X \otimes P$ and P^{ω} morally like $\zeta X.\, X \star P$.

Meta-question (I don't know the answer)

Is this relevant for other applications of containers?

Towards an automata-theoretic problem

We were drawn to studying (total parts) of problems generated by the following closed ζ-expressions:

$$P, Q ::= X \mid \gamma X.P \mid P \,\square\, Q \qquad \square \in \{+, \otimes, \wedge\}, \gamma \in \{\mu, \nu, \zeta\}$$

They're all generated by regular infinite **game-tree** languages.

Game-tree problems

Given an infinite regular tree language

$$L \subseteq \mathsf{Tree}(\{\exists, \forall\} \times \{0, \ldots, k\}) \qquad \text{(a tree = an arena for a parity game)}$$

define the problem $[\![L]\!]$ by $\mathrm{dom}([\![L]\!]) = L$ and

$$[\![L]\!]_G \quad = \quad \text{winning strategies for Eve in } G$$

Most problems we've seen today are equivalent to game-tree problems.

A purely automata-theoretic question?

Conjecture

From automata recognizing L, L', we can decide "$[\![L]\!] \leq_{\mathrm{W}} [\![L']\!]$?".

- Very easy exercise: with $[\![L]\!] \equiv_{\mathrm{W}} \mathbb{1}$
- Maybe exercise (???): with $[\![L']\!] \equiv_{\mathrm{W}} \mathbb{1}$
  - $[\![L]\!] \leq_{\mathrm{W}} \mathbb{1}$ close to "one can continuously turn game trees in L to a winning strategy for the relevant player".
- Not sure about the necessary technology for the more general case

A purely automata-theoretic question?

Conjecture

From automata recognizing L, L', we can decide "$[\![L]\!] \leq_{\mathrm{W}} [\![L']\!]$?".

- Very easy exercise: with $[\![L]\!] \equiv_{\mathrm{W}} \mathbb{1}$
- Maybe exercise (???): with $[\![L']\!] \equiv_{\mathrm{W}} \mathbb{1}$
  - $[\![L]\!] \leq_{\mathrm{W}} \mathbb{1}$ close to "one can continuously turn game trees in L to a winning strategy for the relevant player".
- Not sure about the necessary technology for the more general case
- ($\exists$ a variant of the conjecture in trees with coloured holes in the paper)