The Myhill isomorphism theorem does not generalize much (in a sense)

Cécilia Pradic

July 30th 2026

CCA, Trier

Constructivity (1/3)

Easy observation

$\pi + e$ is transcendental or $e \cdot \pi$ is transcendental (or both are).

Constructivity (1/3)

Easy observation

$\pi + e$ is transcendental or $e \cdot \pi$ is transcendental (or both are).

- I do not know whether $\pi + e$ is transcendental or not...
- nor do I know that for $e \cdot \pi$

Constructivity (1/3)

Easy observation

$\pi + e$ is transcendental or $e \cdot \pi$ is transcendental (or both are).

- I do not know whether $\pi + e$ is transcendental or not...
- nor do I know that for $e \cdot \pi$

Morality

$\rightsquigarrow$ Not all mathematical arguments are equally informative.

Constructivity (2/2)

In broad strokes

Reject excluded middle and reductio ad absurdum.

$$A \vee \neg A \qquad\qquad \neg\neg A \Rightarrow A$$

- Large amounts of mathematics can still be formalized

  (abstract nonsense, finitary combinatorics, $(\mathbb{Q}, <)$)
- Some stuff breaks down

  (analysis, infinitary combinatorics, ordinals, $(\mathbb{R}, <)$)
- Still expressive: classical logic through $\neg\neg$-translation

  (caveat: sets and function spaces not necessarily left untouched)

Some non-constructive axioms

The limited principle of omniscience (LPO)

"For every $p \in 2^{\mathbb{N}}$, either $p = 0^{\omega}$ or $\exists n \in \mathbb{N}.\ p(n) = 1$."

$\sim$ excluded middle for $\boldsymbol{\Sigma^0_1}$ formulas

The lesser limited principle of omniscience (LLPO)

"For every $p \in 2^{\mathbb{N}}$ s.t. $\exists^{\leq 1} k.\ p(k) = 1$,
either $\quad p(2\mathbb{N}) = \{0\} \quad$ or $\quad p(2\mathbb{N}+1) = \{0\}$."

Equivalent statements in analysis (Cauchy reals):

LPO	$\forall x, y \in \mathbb{R}.$ either $x = y$ or $\lvert x - y \rvert \geq 2^{-n}$ for some $n \in \mathbb{N}$
LLPO	$\leq$ is a total order over $\mathbb{R}$: $\forall x, y \in \mathbb{R}.\ x \leq y \vee x \geq y$

A more constructive axiom

Markov's principle (MP)

"For every $p \in 2^{\mathbb{N}}$ such that $p \neq 0^\omega$, $\exists n \in \mathbb{N}.\ p(n) = 1$."

- **Postulated** by **some** constructivists
- Corresponds to unbounded search in realizability models
- LPO $\Rightarrow$ LLPO $\wedge$ MP, separations otherwise

In analysis:

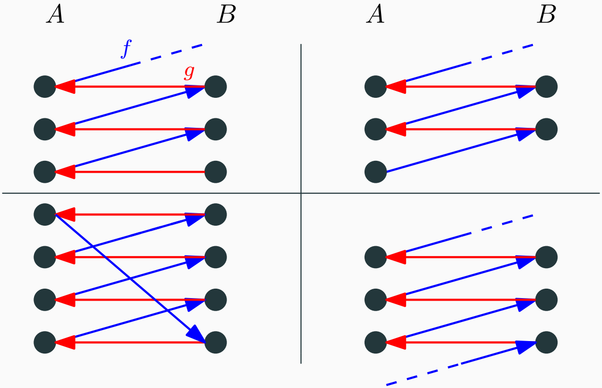
LPO	$\forall x, y \in \mathbb{R}.$ either $x = y$ or $\lvert x - y\rvert \geq 2^{-n}$ for some $n \in \mathbb{N}$
LLPO	$\leq$ is a total order over $\mathbb{R}$: $\forall x, y \in \mathbb{R}.\ x \leq y \vee x \geq y$
MP	$\forall x \in \mathbb{R}.\ x \neq 0 \Rightarrow \exists n \in \mathbb{N}.\ \lvert x\rvert > 2^{-n}$

Some non-classical consistent statements

- All functions $\mathbb{N} \to \mathbb{N}$ are computable.
- All functions $\mathbb{N}^{\mathbb{N}} \to 2$ are continuous.
- All functions $\mathbb{N}^{\mathbb{N}} \to 2$ are Borel and LPO.

The CB theorem

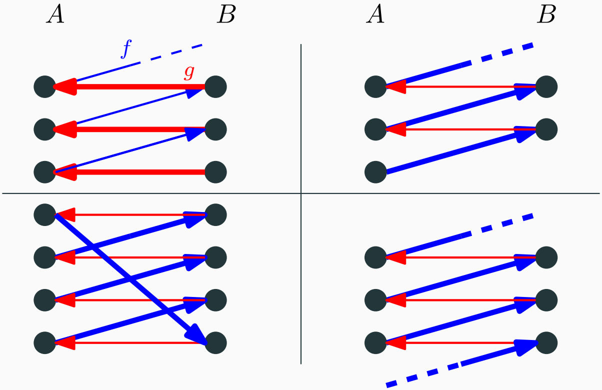
If there exists injections $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$.



Cantor-Bernstein

The CB theorem

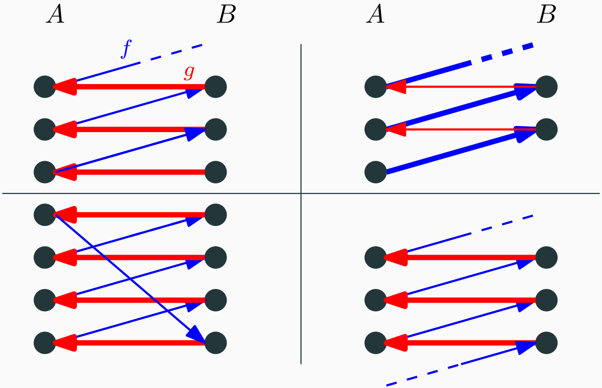
If there exists injections $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$.



Cantor-Bernstein

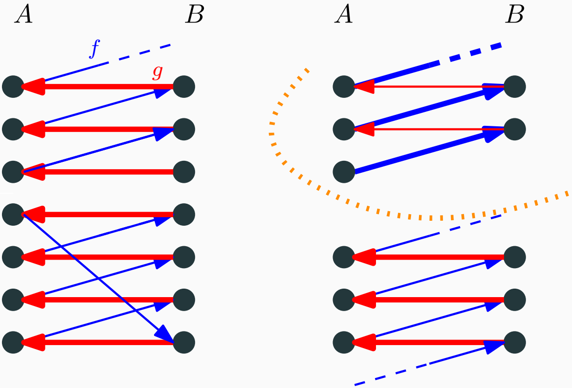
The CB theorem

If there exists injections $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$.



The theorem

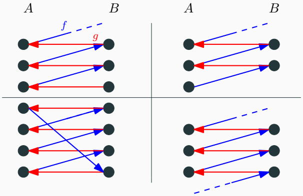
If there exists injections $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$.



$\longrightarrow$ excluded middle used to define h by cases

Why isn't this constructive

- We can ask for the successor of a node in the graph
  - given some $x \in A$, apply f; vice-versa for B and g.
- ...but not predecessor



Main question our function cannot ask

Does my input have a finite and odd number of predecessors?

Failures of Cantor-Bernstein

Idea: adding structure to the map makes CB fail:

Topological and recursion-theoretic failures

- $[0, 1]$ and $(0, 1)$ inject continuously into one another, but aren't homeomorphic!
- $\mathbb{N}$ and the following set computably inject into one another

$$\{e \in \mathbb{N} \mid \text{the } e\text{th Turing machine doesn't halt}\}$$

  but they are not computably isomorphic!

Failures of Cantor-Bernstein

Idea: adding structure to the map makes CB fail:

Topological and recursion-theoretic failures

- $[0,1]$ and $(0,1)$ inject continuously into one another, but aren't homeomorphic!
- $\mathbb{N}$ and the following set computably inject into one another

$$\{e \in \mathbb{N} \mid \text{the } e\text{th Turing machine doesn't halt}\}$$

but they are not computably isomorphic!

Consequence: Cantor-Bernstein fails in a number of models

Failures of Cantor-Bernstein

Idea: adding structure to the map makes CB fail:

Topological and recursion-theoretic failures

- $[0, 1]$ and $(0, 1)$ inject continuously into one another, but aren't homeomorphic!
- $\mathbb{N}$ and the following set computably inject into one another

$$\{e \in \mathbb{N} \mid \text{the } e\text{th Turing machine doesn't halt}\}$$

but they are not computably isomorphic!

Consequence: Cantor-Bernstein fails in a number of models

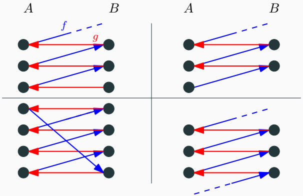
How bad it is?

Banaschewski and Brümmer's reversal

A strengthening of Cantor-Bernstein (CBBB)

If there exists injection $f : A \to B$ and $g : B \to A$, then there exists $h : A \cong B$ *with* $h \subseteq f \cup g^{-1}$

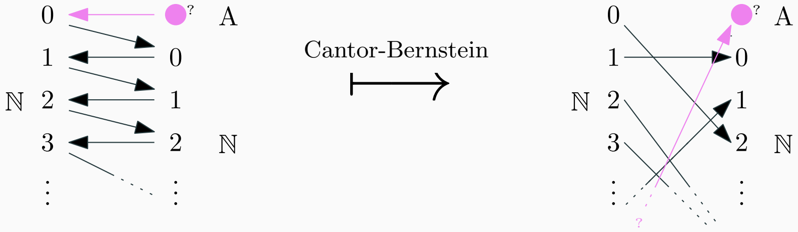
In pictures: we force the bijection to be a subgraph



Theorem (Banaschewski and Brümmer 1986)

Over IZ, CBBB implies excluded middle.

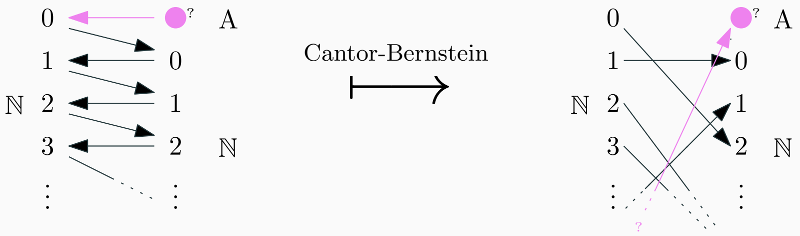
For general Cantor-Bernstein



- $h(0)$ might be uninformative
- But asking "Is $\bullet \in h(\mathbb{N})$?" would be enough

(trivial corollary: CB $\wedge$ LPO $\Rightarrow$ EM)

For general Cantor-Bernstein



- $h(0)$ might be uninformative
- But asking "Is $\bullet \in h(\mathbb{N})$?" would be enough

(trivial corollary: CB $\wedge$ LPO $\Rightarrow$ EM)

Idea

Find some other set $\mathbb{N}_\infty$ for which we can ask our question

"For any $h : \mathbb{N}_\infty \to A \cup \mathbb{N}_\infty$, is $\bullet \in h(\mathbb{N}_\infty)$?"

The conatural numbers $\mathbb{N}_\infty$

Definition as a subset of $2^{\mathbb{N}}$

$$\mathbb{N}_\infty \quad := \quad \{p \in 2^{\mathbb{N}} \mid \exists^{\leq 1} n \in \mathbb{N}.\ p(n) = 1\}$$

- Universal property: final coalgebra for $X \mapsto 1 + X$
- Call ∞ the sequence $n \mapsto 0$
- Embedding $\mathbb{N} \to \mathbb{N}_\infty$: let's write it $n \mapsto \underline{n}$.

The conatural numbers $\mathbb{N}_\infty$

Definition as a subset of $2^{\mathbb{N}}$

$$\mathbb{N}_\infty \quad := \quad \{p \in 2^{\mathbb{N}} \mid \exists^{\leq 1} n \in \mathbb{N}.\ p(n) = 1\}$$

- Universal property: final coalgebra for $X \mapsto 1 + X$
- Call ∞ the sequence $n \mapsto 0$
- Embedding $\mathbb{N} \to \mathbb{N}_\infty$: let's write it $n \mapsto \underline{n}$.
- LPO $\iff$ $\mathbb{N}_\infty = \underline{\mathbb{N}} \cup \{\infty\}$.
- Can constructively define addition, but not subtraction or an equality map $\mathbb{N}_\infty^2 \to 2$

$\mathbb{N}_\infty$ is searchable

Constructive theorem (Escardó 2013)

There is a map $\varepsilon : 2^{\mathbb{N}_\infty} \to \mathbb{N}_\infty$ that picks witnesses

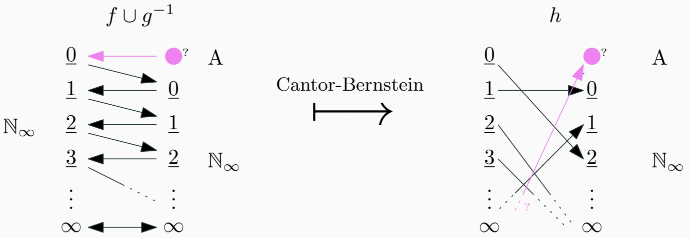
$$\forall p \in 2^{\mathbb{N}_\infty}.\ \ (\exists n \in \mathbb{N}_\infty.\ p(n) = 1) \Longrightarrow p(\varepsilon(p)) = 1$$

Idea: $\varepsilon(p)$ outputs 0s until it finds some $n \in \mathbb{N}$ s.t. $p(\underline{n}) = 1$.

Definition by co-recursion:

$$\varepsilon(p) = \begin{cases} \underline{0} & \text{if } p(\underline{0}) = 1 \\ \underline{\mathrm{Succ}}(\varepsilon(p \circ \underline{\mathrm{Succ}})) & \text{otherwise} \end{cases}$$

Cantor-Bernstein implies excluded middle



- Define $p \in 2^{\mathbb{N}_\infty}$ by $p(n) :=$ "$h(n) = \bullet$"
- Conclude using $p(\varepsilon(p)) = 1 \iff \bullet \in A$

Corollary (Brown, P. 2017)

Cantor-Bernstein implies excluded middle.

An even easier exercise, and a question

The dual Cantor-Bernstein?

The following statement implies excluded middle:

"If there are surjections $A \twoheadrightarrow B$ and $B \twoheadrightarrow A$, then there is an injection $A \hookrightarrow B$"

Similar but different sophistry. (does not involve $\mathbb{N}_\infty$)

An even easier exercise, and a question

The dual Cantor-Bernstein?

The following statement implies excluded middle:

"If there are surjections $A \twoheadrightarrow B$ and $B \twoheadrightarrow A$, then there is an injection $A \hookrightarrow B$"

Similar but different sophistry. (does not involve $\mathbb{N}_\infty$)

Hint: quotient $\mathbb{N}$

Question,s assuming EM (Banaschewski & Moore 90, . . .)

Is the dual Cantor-Bernstein equivalent to AC?

(or the (weak) partition principle?)

Is this actually informative?

The arguments rely on making one of the set horrible dependent on some arbitrary proposition we want to decide.

- Gives only lousy concrete counter-examples in non 2-valued models (afaik)
- Does not speak to what we could know if we limit the complexity of A, B, f and g...

The Myhill isomorphism theorem

Reduction

$A \subseteq \mathbb{N}$ **reduces** to $B \subseteq \mathbb{N}$ **via** $f : \mathbb{N} \to \mathbb{N}$ iff $f^{-1}(B) = A$.

Constructive theorem (Myhill 1955)

If $A, B \subseteq \mathbb{N}$ are inter-reducible via injections $\mathbb{N} \to \mathbb{N}$, then there exists a bijection $h : \mathbb{N} \to \mathbb{N}$ with $h(A) = B$.

- Official original version: insert two "computable" above
- A and B could be **arbitrarily horrible**

The Myhill isomorphism theorem

Reduction

$A \subseteq \mathbb{N}$ **reduces** to $B \subseteq \mathbb{N}$ **via** $f : \mathbb{N} \to \mathbb{N}$ iff $f^{-1}(B) = A$.

Constructive theorem (Myhill 1955)

If $A, B \subseteq \mathbb{N}$ are inter-reducible via injections $\mathbb{N} \to \mathbb{N}$, then there exists a bijection $h : \mathbb{N} \to \mathbb{N}$ with $h(A) = B$.

- Official original version: insert two "computable" above
- A and B could be **arbitrarily horrible**

$\Rightarrow$ h can be built only with info from the injections

Towards a proof of the Myhill isomorphism theorem

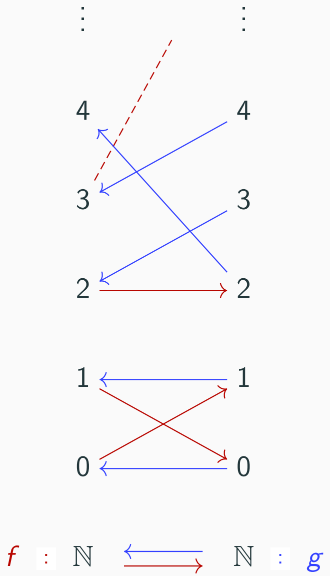
Let's call this the strong Myhill isomorphism theorem

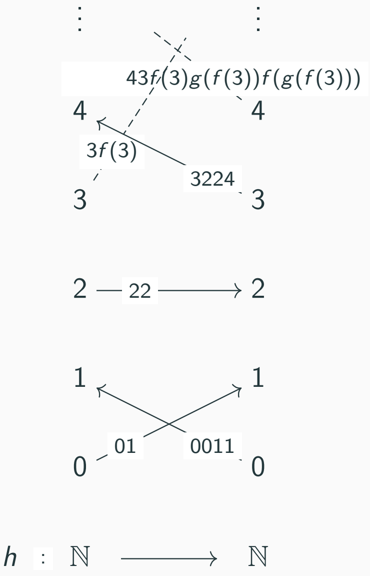
Given two injections $f, g : \mathbb{N} \to \mathbb{N}$, $\exists$ a bijection $h : \mathbb{N} \to \mathbb{N}$ s.t.

$$h \subseteq \bigcup_{m \in \mathbb{Z}} (f \circ g)^m \circ f$$

- Compare and contrast with CBBB (when both sets are $\mathbb{N}$):
  - CBBB says $h \subseteq f \cup g^{-1}$ $(m \in \{-1, 0\})$
  - Pictures: we can only use **edges** in the graph given by f and g
  - Relaxation: we can use **paths**
- Implies the Myhill isomorphism theorem
  - If f, g are reductions between A and B, then the connected components are either in $A + B$ or outside.

Proof: a back-and-forth argument





Question: replacing $\mathbb{N}$? (Bauer 2025, fediverse)

Definition

Say that X has the **Myhill property** if:

For all $A, B \subseteq X$ are inter-reducible via injections,
there exists a bijection $h : X \to X$ with $h(A) = B$.

Questions

Is/does the class of sets with the Myhill property

1. closed under $+, \times, \to$?
2. contain $\mathbb{N}_\infty$?

(constructively; classically, that's a corollary of CBBB)

Before we discuss this

Strong Myhill property: defined analogously

Definition

Say that X has the **strong** Myhill property if:

For any injections $f, g : X \to X$

there exists a bijection $h : X \to X$ with $h \subseteq \bigcup_{m \in \mathbb{Z}} (f \circ g)^m \circ f$.

- Clearly implies the Myhill property.
- Converse: not clear (to me).

Closure under $+, \times, \to$ is not reasonable

Observation (†)

For $n \in \mathbb{N}$, any $A \subseteq \{0, \ldots, n\}$ has the strong Myhill property.

Proof: $g^{-1} = (f \circ g)^{n!-1} \circ f$

Closure under $+, \times, \to$ is not reasonable

Observation (†)

For $n \in \mathbb{N}$, any $A \subseteq \{0, \ldots, n\}$ has the strong Myhill property.

Proof: $g^{-1} = (f \circ g)^{n!-1} \circ f$

Corollary of (†) and the Myhill isomorphism theorem

LPO and the closure of the Myhill property under either $+, \times, \to$ or subsets imply excluded middle.

Proof idea: essentially the same as CBBB $\wedge$ LPO $\Rightarrow$ EM

$\mathbb{N}_\infty$ does not have the Myhill property

- Assume $\mathbb{N}_\infty$ has the strong Myhill property
- Assume $\mathbb{N}_\infty$-choice: every surjection $A \to \mathbb{N}_\infty$ has a section
- (valid in Kleene-Vesley realizability)

Straightforward consequence of all of that

For injections $f, g : \mathbb{N}_\infty \to \mathbb{N}_\infty$, there is $\iota : \mathbb{N}_\infty \to \mathbb{Z}$ such that

$$h(x) = (f \circ g)^{\iota(x)}(f(x)) \quad \text{is a bijection}$$

(ι tells us how to travel in the graph to define h)

$\mathbb{N}_\infty$ does not have the Myhill property

- Assume $\mathbb{N}_\infty$ has the strong Myhill property
- Assume $\mathbb{N}_\infty$-choice: every surjection $A \to \mathbb{N}_\infty$ has a section
- (valid in Kleene-Vesley realizability)

Straightforward consequence of all of that

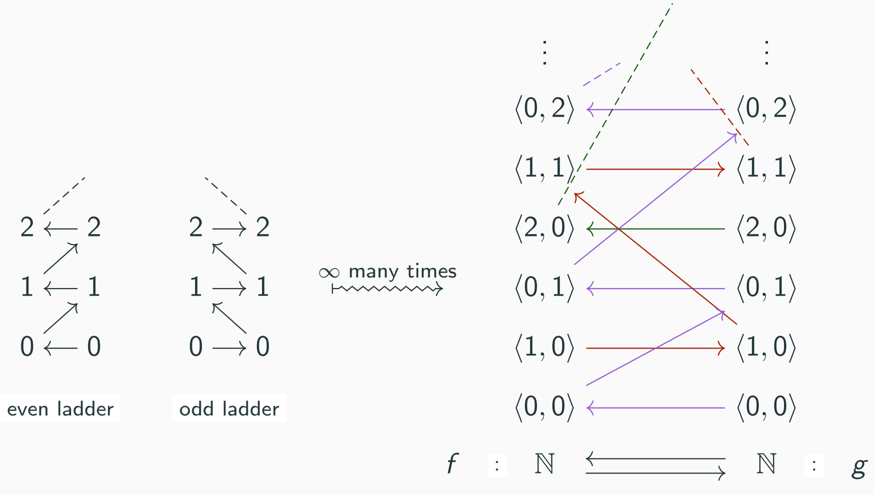
For injections $f, g : \mathbb{N}_\infty \to \mathbb{N}_\infty$, there is $\iota : \mathbb{N}_\infty \to \mathbb{Z}$ such that

$$h(x) = (f \circ g)^{\iota(x)}(f(x)) \quad \text{is a bijection}$$

(ι tells us how to travel in the graph to define h)

$\iota : \mathbb{N}_\infty \to \mathbb{Z}$ is continuous iff it is eventually constant.

Forcing ι to oscillate between positive and negative (boom)



Formally

Theorem

If $\mathbb{N}_\infty$ has the strong Myhill property, MP holds and $\mathbb{N}_\infty$-choice holds, then LPO holds.

Technical lemma, in Kleene-Vesley realizability

If X is a partitioned modest set and has the Myhill property, then it has the strong Myhill property.

Proof: given f and g, make $A, B \subseteq \mathbb{N}_\infty$ horrible enough.

Theorem

$\mathbb{N}_\infty$ does not have the Myhill property in KV realizability.

But...

- We have *not really* shown that a reasonable bijection is impossible to build from f and g alone.
- Only that it is not induced by a continuous $\iota : \mathbb{N}_\infty \to \mathbb{Z}$

Fix by inserting $\neg\neg$

Say that X has the **strong** $\neg\neg$-Myhill property if:

For any injections $f, g : X \to X$

there exists a bijection $h : X \to X$ such that

$\neg\neg(\exists m \in \mathbb{Z}.\ h(x) = (f \circ g)^m(f(x)))$ for every $x \in X$

But. . .

- We have *not really* shown that a reasonable bijection is impossible to build from f and g alone.
- Only that it is not induced by a continuous $\iota : \mathbb{N}_\infty \to \mathbb{Z}$

Fix by inserting $\neg\neg$

Say that X has the **strong** $\neg\neg$-Myhill property if:

For any injections $f, g : X \to X$

there exists a bijection $h : X \to X$ such that

$\neg\neg(\exists m \in \mathbb{Z}.\ h(x) = (f \circ g)^m(f(x)))$ for every $x \in X$

Theorem

If MP holds, $\mathbb{N}_\infty$ has the strong $\neg\neg$-Myhill property.

Very rough proof idea

Assume $f, g : \mathbb{N}_\infty \to \mathbb{N}_\infty$ injective.

Observation

If f, g are continuous, $f(\infty) = g(\infty) = \infty$

Start an optimistic back-and-forth on the elements $< \infty$

- If we need the value of $f(\underline{n})$, actually query $\min(f(\infty), f(\underline{n}))$.
- If $\min(f(\infty), f(\underline{n})) = f(\infty)$, f is discontinuous and LPO holds $\implies$ we have $\mathbb{N}_\infty \cong \mathbb{N}$ (all becomes easy)
- Otherwise $f(\underline{n}) < \infty$; we're happy and we carry on.
- (completely analogous for g queries)

Some subtleties, but h can be built from that and the $\neg\neg$ in the correctness criterion allows the use of classical logic there.

The $\neg\neg$-Myhill property beyond $\mathbb{N}_\infty$?

Strong counter-examples

If MP holds and any of

$$\mathbb{N} + \mathbb{N}_\infty \quad \mathbb{N} \times \mathbb{N}_\infty \quad \mathbb{N}_\infty^2 \quad 2^{\mathbb{N}} \quad \text{or} \quad \mathbb{N}^{\mathbb{N}}$$

have the strong $\neg\neg$-Myhill property, then LPO holds.

Boils down to finding easy injections f, g such that no continuous bijection h can do the job.

Remaining conjecture for converses (easy?)

$2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ have the property $\Longrightarrow$ $\boldsymbol{\Sigma}_1^1$-excluded middle.

The ¬¬-Myhill property beyond $\mathbb{N}_\infty$?

Strong counter-examples

If MP holds and any of

$$\mathbb{N} + \mathbb{N}_\infty \quad \mathbb{N} \times \mathbb{N}_\infty \quad \mathbb{N}_\infty^2 \quad 2^{\mathbb{N}} \quad \text{or} \quad \mathbb{N}^{\mathbb{N}}$$

have the strong ¬¬-Myhill property, then LPO holds.

Boils down to finding easy injections f, g such that no continuous bijection h can do the job.

Remaining conjecture for converses (easy?)

$2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ have the property $\Longrightarrow$ $\boldsymbol{\Sigma}^1_1$-excluded middle.

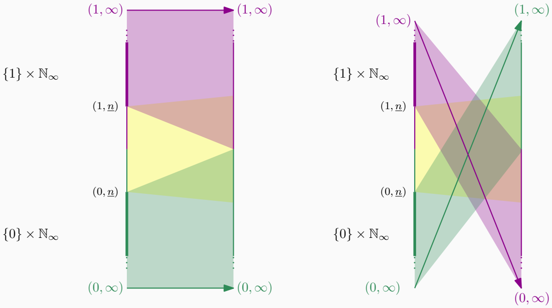
Missing $k \times \mathbb{N}_\infty$ for $k \in \mathbb{N} \setminus \{0, 1\}$?

$2 \times \mathbb{N}_\infty$: h can be continuous

A positive result

Assuming LPO, given uniformly continuous injections $f, g : 2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$, there exists a continuous bijection $h : 2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$ such that $h \subseteq \bigcup_{m \in \mathbb{Z}} (f \circ g)^m \circ f$.

B/c continuous injections $2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$ look like that:



$2 \times \mathbb{N}_\infty$: h cannot be continuously computed from f and g

Theorem

In KV realizability, $2 \times \mathbb{N}_\infty$ does **not** have the $\neg\neg$-Myhill property.

$2 \times \mathbb{N}_\infty$: h cannot be continuously computed from f and g

Theorem

In KV realizability, $2 \times \mathbb{N}_\infty$ does **not** have the $\neg\neg$-Myhill property.

Quantifying the obstruction via modalities

For any two injections $f, g : 2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$, there $\mathrm{LLPO}^* \star \mathrm{LPO}^8$-exists a suitable bijection h such that $\forall x \in 2 \times \mathbb{N}_\infty.\ \bigcirc_{\mathrm{LPO}} (\exists m \in \mathbb{Z}.\ h(x) = (f \circ g)^m(f(x)))$.

- LPO^8 can be dropped when f and g are continuous
- Plausible conjecture: then LLPO^* is optimal

What does it mean?

Modalities from problems (Weihrauch or otherwise)

Given an $F : I \to \mathcal{P}(O)$, define (wlog $O = 1$)

$$\begin{array}{rccc} \bigcirc_F : & \Omega & \longrightarrow & \Omega \\ & \varphi & \longmapsto & \exists i \in I.\, \forall o \in F(i).\, \varphi \end{array}$$

- Intuition for proving $\bigcirc_F \varphi$: if someone has an answer to a F-question of my choosing, I can prove φ.
- number of other sanity checks can be made

$$\bigcirc_F \varphi \wedge (\forall i \in I. \exists o \in F(i)) \Rightarrow \varphi \qquad \forall i \in I. \bigcirc_F (\exists o \in F(i)) \qquad \ldots$$

Only one call!

$\bigcirc_F \bigcirc_F \varphi \not\Rightarrow \bigcirc_F \varphi$ in general

Context

Observation (Kihara 21, ...)

(extended Weihrauch) problems F

$$\iff$$

monotone $j : \Omega \to \Omega$ in (Kleene-Vesley) realizability

- $\varphi \Rightarrow \bigcirc_F \varphi$ is equivalent to F pointed
- $\bigcirc_F \bigcirc_F \varphi \Rightarrow \bigcirc_F \varphi$ = closure under $\star/\circ$

$\Rightarrow$ = Lawvere-Tierney topologies += closure under $\star/\circ$

- One may generically also talk about parallel combinations, ...

Generic question

Use that for more fine-grained reverse maths?

(can also be formalized in e.g. subsystems of Z_2, ...)

Modalities for sets

Definition

Given an $F : I \to \mathcal{P}(O)$, define

$$\begin{array}{rccc} \bigcirc_F : & \mathrm{Set} & \longrightarrow & \mathrm{Set} \\ & X & \longmapsto & \{f : F(i) \to X \mid f \text{ constant}, i \in I\}_{/\sim} \end{array}$$

- Having an $\tilde{x} \in \bigcirc_F X$: should you be able to solve an arbitrary F-challenge, you can get an $x \in X$!
- (any solution $\to$ same result)
- (identify things that ultimately yield the same $x \in X$)
- Modalities: functorial action on injections into 1.

Another example

The problem $\mathsf{C}_{\omega+1,2}$

- **Input:** a decreasing sequence $s \in (\omega+1)^\omega$
- **Output:** $b \in 2$ equal to the parity of $\min(s)$ if $\min(s) \neq \omega$

Call η the canonical map $2^{\mathbb{N}} \to \bigcirc_{\mathsf{C}_{\omega+1,2}}(2^{\mathbb{N}})$

CBBB for $2^{\mathbb{N}}$ and continuous maps (Neumann, Pauly, P.)

In KV-realizability, for any injections $f, g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, there is a "bijection" $h : 2^{\mathbb{N}} \to \bigcirc_{\mathsf{C}_{\omega+1,2}}(2^{\mathbb{N}})$ such that, for every $p \in 2^{\mathbb{N}}$,

$$\bigcirc_{\mathsf{C}_{\omega+1,2}} \left(h(x) = \eta(f(x)) \quad \vee \quad h(x) = \eta(g^{-1}(x)) \right)$$

Back to this Myhill property: summary (assuming MP)

- For operators:

$$(\text{Closure under } +, \times, \to) \quad \Longrightarrow \quad \text{excluded middle}$$

- For simple sets:

... having the $\neg\neg$-Myhill property	is equivalent to ...
$\mathbb{N}$ subfinite sets $\mathbb{N}_\infty$	$\top$
$\mathbb{N}_\infty \times 2$ $\mathbb{N}_\infty \times 3$ ...	? $\in$ [LLPO, LPO]
$\mathbb{N} + \mathbb{N}_\infty$ $\mathbb{N} \times \mathbb{N}_\infty$ $\mathbb{N}_\infty^2$	LPO
$2^{\mathbb{N}}$ $\mathbb{N}^{\mathbb{N}}$	$\mathbf{\Sigma_1^1}$ – EM?

Some takeaways

- KV realizability useful for intuitions!
- As well as oracle modalities/functors
    - can be used in a model-agnostic way in the logic
    - connecting Weihrauch complexity to higher-order problems
- Frivolous, but reasonably fun??
- Does not speak much to other CB-flavored works out there?

(Gowers 1996, Goodrick 2001, ...)

Some questions

- Framing in higher-order reverse maths?
- What is the complexity of $\neg\neg$-CBBB for $\mathbb{N}$? $\mathbb{N}_\infty$? $k \times \mathbb{N}_\infty$?
- Can a univalent universe have the Myhill property?

  (not sure if that was one of the questions of Andrej)
- Can we say something about "set divison" theorems?

$$X \times k \cong Y \times k \implies X \cong Y \qquad (k \in \mathbb{N})$$

Some questions

- Framing in higher-order reverse maths?
- What is the complexity of $\neg\neg$-CBBB for $\mathbb{N}$? $\mathbb{N}_\infty$? $k \times \mathbb{N}_\infty$?
- Can a univalent universe have the Myhill property?

  (not sure if that was one of the questions of Andrej)
- Can we say something about "set divison" theorems?

$$X \times k \cong Y \times k \implies X \cong Y \qquad (k \in \mathbb{N})$$

Thanks for listening! Questions? :)

Modalities in action

$\mathsf{LPO}(p) = \{n+1 \mid p(n) = 1\} \cup \{0 \mid p = 0^\omega\}$ ...

Memento $2 \times \mathbb{N}_\infty$

For any two injections $f, g : 2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$, there $\mathsf{LLPO}^* \star \mathsf{LPO}^8$-exists a suitable bijection h such that $\forall x \in 2 \times \mathbb{N}_\infty.\ \bigcirc_{\mathsf{LPO}} (\exists m \in \mathbb{Z}.\ h(x) = (f \circ g)^m(f(x)))$.

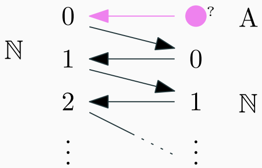
Banaschewski and Brümmer's reversal (2/2)

Theorem (Banaschewski and Brümmer 1986)

Over IZ, CBBB implies excluded middle.

Fix $A \subseteq \{\bullet\}$ and build maps $f : \mathbb{N} \to A \cup \mathbb{N}$ and $g : A \cup \mathbb{N} \to \mathbb{N}$

$$f(n) := n \qquad g(\bullet) := 0 \qquad g(n) := n + 1$$



Is A inhabited?

$\to$ is $h(0) = \bullet$ or 0?

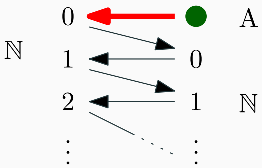
Banaschewski and Brümmer's reversal (2/2)

Theorem (Banaschewski and Brümmer 1986)

Over IZ, CBBB implies excluded middle.

Fix $A \subseteq \{\bullet\}$ and build maps $f : \mathbb{N} \to A \cup \mathbb{N}$ and $g : A \cup \mathbb{N} \to \mathbb{N}$

$$f(n) := n \qquad g(\bullet) := 0 \qquad g(n) := n + 1$$



Is A inhabited?

$\to$ is $h(0) = \bullet$ or 0?

Yes!

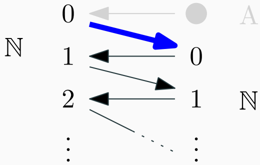
Banaschewski and Brümmer's reversal (2/2)

Theorem (Banaschewski and Brümmer 1986)

Over IZ, CBBB implies excluded middle.

Fix $A \subseteq \{\bullet\}$ and build maps $f : \mathbb{N} \to A \cup \mathbb{N}$ and $g : A \cup \mathbb{N} \to \mathbb{N}$

$$f(n) := n \qquad g(\bullet) := 0 \qquad g(n) := n+1$$



Is A inhabited?

$\to$ is $h(0) = \bullet$ or 0?

No!