Synthesizing nested relational queries from implicit specifications

Cécilia Pradic (Swansea University)
j.w.w. Michael Benedikt (Oxford University) and Christoph Wernhard (TU Dresden)
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Extracting programs from specifications

The general setting

Input: a *logical* input-output specification $\varphi(i, o)$
Output: a *program* `f` obeying the specification

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- **Functionality**: at most one output per input $\forall i. \forall o. \forall o'. \varphi(i, o) \wedge \varphi(i, o') \Rightarrow o = o'$

Extracting programs from specifications and proofs

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- ▶ produce an extension $\mathtt{f}$ s.t. $\forall i.\forall o.\ \varphi(i, o) \Rightarrow \mathtt{f}(i) = o$

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- Main result: implicit $\rightarrow$ explicit NRC definitions
  - One easy method for intuitionistic definitions
  - One harder method for classical definitions
  - Linear-time on *cut-free focused* proofs

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- Main result: implicit $\to$ explicit NRC definitions
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  - Linear-time on *cut-free focused* proofs
- WIP: generalization to effective rigid categoricity

The nested relational model, logic and NRC

The nested relational model

We work with **typed objects**

Types for nested collections

$$T, U ::= \mathfrak{U} \mid \mathsf{Set}(T) \mid 1 \mid T \times U$$

Anonymous base type $\mathfrak{U}$
Semantics $T \mapsto [\![T]\!]$ determined inductively by $[\![\mathfrak{U}]\!]$:

- $\mathsf{Set}(T)$: sets of elements of type T
- finite cartesian products $- \times \ldots \times -$

Examples

Taking $[\![\mathfrak{U}]\!] = \texttt{string}$, we have

$$\begin{array}{lcl} \{(``\texttt{snake}", ``\texttt{slange}"), (``\texttt{pencil}", ``\texttt{blyant}"), \ldots\} & \in & [\![\mathsf{Set}(\mathfrak{U} \times \mathfrak{U})]\!] \\ \{(\{``\texttt{snake}", ``\texttt{serpent}"\}, \{``\texttt{slange}", ``\texttt{snog}"\}), \ldots\} & \in & [\![\mathsf{Set}(\mathsf{Set}(\mathfrak{U}) \times \mathsf{Set}(\mathfrak{U}))]\!] \\ ((), \emptyset, ``\texttt{snake}", \{``\texttt{slange}", ``\texttt{snog}"\}) & \in & [\![1 \times \mathsf{Set}(\mathsf{Set}(1)) \times \mathfrak{U} \times \mathsf{Set}(\mathfrak{U})]\!] \end{array}$$

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Usual relational model: only tuples of relations (sets of tuples)

Types for nested collections

$$T, U ::= \mathfrak{U} \mid \mathsf{Set}(T) \mid 1 \mid T \times U$$

A transformation of nested sets is a function $T \to U$

$\to$ is **not** part of the type system

A transformation of flat relations

Pre-image of a relation R

$$\begin{array}{rccl} \mathsf{fib} : & \mathsf{Set}(\mathfrak{U}) \times \mathsf{Set}(\mathfrak{U} \times \mathfrak{U}) & \to & \mathsf{Set}(\mathfrak{U}) \\ & (A, R) & \mapsto & R^{-1}(A) = \{x \mid \exists y \in A.(x, y) \in R\} \end{array}$$

A transformation of nested collections

Collect all pre-images of individual elements

$$\begin{array}{rccl} \mathsf{fibs} : & \mathsf{Set}(\mathfrak{U} \times \mathfrak{U}) & \to & \mathsf{Set}(\mathfrak{U} \times \mathsf{Set}(\mathfrak{U})) \\ & R & \mapsto & \{(a, \mathsf{fib}(\{a\}, R)) \mid a \in \mathsf{cod}(R)\} \end{array}$$

Logical specfications

Queries can be specified in *multi-sorted* first-order logic:

- variables explicitly typed $x : T$
- basic predicates $x \in_T z$ and $x =_T y$ $\quad$ $x, y : T$ and $z : \mathsf{Set}(T)$
- terms for tupling and projections $\quad$ e.g., $\pi_1(((x, z), ()), x)) : (T \times \mathsf{Set}(T)) \times 1$

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Consider formulas with only bounded quantifications

Δ_0 formulas

$$\varphi, \psi \quad ::= \quad t =_T u \mid t \in_T u \mid \exists x \in t\ \varphi \mid \forall x \in t\ \varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi$$

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Example of **functional** and **total** specifications:

$\varphi_{\mathsf{fib}}(A, R, X)$ for $X = R^{-1}(A)$

- Every $x \in X$ is related to some $a \in A$ — $\forall x \in X.\exists a \in A.\ (x, a) \in R$
- For every $(x, y) \in R$, if $y \in A$, then $x \in X$ — $\forall p \in R.\ \pi_2(p) \in A \Rightarrow \pi_1(p) \in X$

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Example of **functional** and **total** specifications:

$\varphi_{\mathsf{fibs}}(R, O)$ for $O = \{(a, R^{-1}(\{a\})) \mid a \in \mathsf{cod}(R)\}$

- For every $(x, a) \in R$, there is some $(a, X) \in O$ s.t. $x \in X$

  $\forall p \in R. \exists q \in O.\ \pi_1(p) \in \pi_2(O)$

- Every element of $(a, X) \in O$ satisfies $\varphi_{\mathsf{fib}}(\{a\}, R, X)$

  $\forall q \in O.\ (\forall x \in \pi_2(q).(x, \pi_1(q)) \in R) \wedge (\forall p \in R.\ \pi_2(p) = \pi_1(q) \Rightarrow \pi_1(p) \in \pi_2(q))$

Our target for synthesis: the nested relational calculus (NRC)

Our programming language for nested transformations $\Gamma \to T$

$$\frac{}{\Gamma,\ x : T,\ \Gamma' \vdash x : T}$$

$$\frac{}{\Gamma \vdash () : 1} \qquad \frac{\Gamma \vdash e_1 : T_1 \qquad \Gamma \vdash e_2 : T_2}{\Gamma \vdash \langle e_1, e_2 \rangle : T_1 \times T_2} \qquad \frac{\Gamma \vdash e : T_1 \times T_2 \qquad i \in \{1,2\}}{\Gamma \vdash \pi_i(e) : T_i}$$

$$\frac{\Gamma \vdash e : T}{\Gamma \vdash \{e\} : \mathsf{Set}(T)} \qquad \frac{\Gamma \vdash e_1 : \mathsf{Set}(T_1) \qquad \Gamma,\ x : T_1 \vdash e_2 : \mathsf{Set}(T_2)}{\Gamma \vdash \bigcup \{e_2 \mid x \in e_1\} : \mathsf{Set}(T_2)}$$

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Expressiveness of NRC

Our running examples

- $(A, R) \mapsto \bigcup\{\mathsf{case}(\pi_2(p) \in_{\mathfrak{U}} A, \{\pi_1(p)\}, \emptyset) \mid p \in R\}$
- $R \mapsto \bigcup\{\{\mathsf{fib}(x, R)\} \mid x \in \{\pi_1(p) \mid p \in R\}\}$

Derivable constructs:

- maps $\{e_1(x) \mid x \in e_2\}$
- at type-level, $\mathsf{Bool} := \mathsf{Set}(1)$
- basic predicates $=_T : T \times T \to \mathsf{Bool}$, $\in_T : T \times \mathsf{Set}(T) \to \mathsf{Bool}$
- case analyses

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- case analyses
- Δ_0-separation $\{x \in e \mid \varphi(x)\}$

Proposition

NRC terms $e : T \to \mathsf{Bool}$ correspond exactly to Δ_0 formulas $\varphi(x^T)$

Extraction from Δ_0 specifications

Our effective result

Recall that $\varphi(i, o)$ is an implicit definition when it is functional: $\varphi(i, o) \wedge \varphi(i, o') \implies o = o'$

Extraction from Δ_0 intuitionistic implicit definitions

For every such $\varphi(i, o)$, there is a compatible NRC term $e(i)$

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Nota Bene

- Effectivity w/o efficiency: follows from completeness, compactness and an easy NRC/logical interpreatation correspondence
  - Efficiency is the ultimate goal
- Extension of Beth definability for flat queries $\mathsf{Set}(\mathfrak{U}^k) \times \ldots \times \mathsf{Set}(\mathfrak{U}^m) \to \mathsf{Set}(\mathfrak{U}^n)$
  - Can give some ideas for lower bounds

Potential use-cases for implicit→explicit

Use-case #1: inverting a transformation

Consider an *injective* NRC term such as fibs

$$\begin{array}{rccc} \mathsf{fibs}: & \mathsf{Set}(\mathfrak{U} \times \mathfrak{U}) & \to & \mathsf{Set}(\mathfrak{U} \times \mathsf{Set}(\mathfrak{U})) \\ & R & \mapsto & \{(a, R^{-1}(a)) \mid a \in \mathsf{cod}(f)\} \end{array}$$

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- $\varphi(R, G)$ defines a partial function $G \mapsto R$

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Use-case #2: views

Assume an imperative extension and a program

$$x := e_1(i); \ldots; y := e_2(i)$$

When e_2 is functional in terms of e_1:

- Compute $e_2'(x)$ from a proof

⇝ Potential optimization if $e_1(i)$ is costly?

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Caveat: automation for functionality proofs?

Toward a proof system for nested sets

Wlog, we restrict to the following syntax

$$\begin{array}{lcl} t, u & ::= & x \mid (t, u) \mid \pi_1(t) \mid \pi_2(t) \mid () \\ \varphi, \psi & ::= & t =_{\mathfrak{U}} u \mid t \neq_{\mathfrak{U}} u \mid \exists x \in_T t\ \varphi \mid \forall x \in_T t\ \varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \end{array}$$

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Derived formulas

$$
\begin{array}{lcl lcl}
t \in_T u & := & \exists x \in u.\, t =_T u & t \subseteq_T u & := & \forall x \in t.\, x \in_T u \\
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\end{array}
$$

- Bakes the axiom of extensionality in the definition of $=_T$
- **No further set-theoretic axioms**

What is a proof

Straightforward variants of the sequent calculus

- Sequents $\Gamma \vdash \Delta$ with Γ, Δ lists of Δ_0 formulas
- Intended semantics: $\bigwedge_{\phi \in \Gamma} \phi \implies \bigvee_{\psi \in \Delta} \psi$
- Deduction according to proof rules of the shape

$$\frac{\Gamma_1 \vdash \Delta_1 \qquad \dots \qquad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

Left and right rules for each connectives + *structural rules* + *cut*

Examples

$$\exists\text{-R}\ \frac{\Gamma, t \in u \vdash \phi[t/x], \Delta}{\Gamma, t \in u \vdash \exists x \in u.\phi, \Delta} \qquad \textsc{cut}\ \frac{\Gamma \vdash \phi, \Delta \qquad \Gamma, \phi \vdash \Delta}{\Gamma \vdash \Delta} \qquad \textsc{axiom}\ \frac{}{\Gamma, \phi \vdash \phi, \Delta}$$

Certificate that $\varphi(i, o)$ is an implicit definition: a derivation

$$\cdot;\ \varphi(i, o),\ \varphi(i, o') \vdash o = o'$$

$$\text{AX}\ \text{- - - - - - - - - - - - - - - - - - -}\ (7)$$

$$z \in o,\ x \in X,\ z \in x;\ z \in o' \vdash z \in o'$$

$$\Rightarrow\text{-L}\ \text{- -}\ (6)$$

$$z \in o,\ x \in X,\ z \in x;\ \chi(X, x, z),\ \chi(X, x, z) \Rightarrow z \in o' \vdash z \in o'$$

$$\forall\text{-L}\ \text{———————————————}\ (5)$$

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$$\text{=-SUBST}\ \text{———————————————}$$

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$$z \in o,\ x \in X;\ \psi(X, x, z),\ \Sigma(X, o') \vdash z \in o'$$

$$\exists\text{-L}\ \text{———————————————}\ (3)$$

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$$z \in o;\ \Sigma(X, o),\ \Sigma(X, o') \vdash z \in o'$$

$$\subseteq\text{-R}\ \text{———————————————}\ (2)$$

$$\cdot;\ \Sigma(X, o),\ \Sigma(X, o') \vdash o \subseteq o'$$

$$=_{\mathsf{Set}}\text{-R}\ \text{- - - - - - - - - - - - - - -}\ (1)$$

$$\cdot;\ \Sigma(X, o),\ \Sigma(X, o') \vdash o = o'$$

Certificate that $\varphi(i, o)$ is an implicit definition: a derivation

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$$\text{AX (7)}: \quad z \in o,\ x \in X,\ z \in x;\ z \in o' \vdash z \in o'$$

$$\Rightarrow\text{-L (6)}: \quad z \in o,\ x \in X,\ z \in x;\ \chi(X,x,z),\ \chi(X,x,z) \Rightarrow z \in o' \vdash z \in o'$$

$$\forall\text{-L (5)}: \quad z \in o,\ x \in X,\ z \in x;\ \chi(X,x,z),\ \forall a \in x\ (\chi(X,x,a) \Rightarrow a \in o') \vdash z \in o'$$

$$=\text{-SUBST}: \quad z \in o,\ x \in X,\ z' \in x;\ z =_{\mathcal{U}} z',\ \chi(X,x,z),\ \forall a \in x\ (\chi(X,x,a) \Rightarrow a \in o') \vdash z \in o'$$

$$\exists\text{-L}: \quad z \in o,\ x \in X;\ z \in x,\ \chi(X,x,z),\ \forall a \in x\ (\chi(X,x,a) \Rightarrow a \in o') \vdash z \in o'$$

$$\wedge\text{-L}: \quad z \in o,\ x \in X;\ \psi(X,x,z),\ \forall a \in x\ (\chi(X,x,a) \Rightarrow a \in o') \vdash z \in o'$$

$$\forall\text{-L (4)}: \quad z \in o,\ x \in X;\ \psi(X,x,z),\ \forall y \in X\ \forall a \in y\ (\chi(X,y,a) \Rightarrow a \in o') \vdash z \in o'$$

$$\wedge\text{-L}: \quad z \in o,\ x \in X;\ \psi(X,x,z),\ \Sigma(X,o') \vdash z \in o'$$

$$\exists\text{-L (3)}: \quad z \in o;\ \exists x \in X\ \psi(X,x,z),\ \Sigma(X,o') \vdash z \in o'$$

$$\forall\text{-L}: \quad z \in o;\ \forall a \in o\ \exists x \in X\ \psi(X,x,a),\ \Sigma(X,o') \vdash z \in o'$$

$$\wedge\text{-L}: \quad z \in o;\ \Sigma(X,o),\ \Sigma(X,o') \vdash z \in o'$$

$$\subseteq\text{-R (2)}: \quad \cdot;\ \Sigma(X,o),\ \Sigma(X,o') \vdash o \subseteq o'$$

$$=_{\mathsf{Set}}\text{-R (1)}: \quad \cdot;\ \Sigma(X,o),\ \Sigma(X,o') \vdash o = o'$$

Proof idea for efficient extraction: compute an explicit definition by induction over the proof

Certificate that $\varphi(i, o)$ is an implicit definition: a derivation

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$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{\textsc{ax}}
\dashedLine
\RightLabel{(7)}
\UnaryInfC{$z \in o,\ x \in X,\ z \in x;\ z \in o' \vdash z \in o'$}
\LeftLabel{$\Rightarrow$-L}
\dashedLine
\RightLabel{(6)}
\UnaryInfC{$z \in o,\ x \in X,\ z \in x;\ \chi(X, x, z),\ \chi(X, x, z) \Rightarrow z \in o' \vdash z \in o'$}
\LeftLabel{$\forall$-L}
\RightLabel{(5)}
\UnaryInfC{$z \in o,\ x \in X,\ z \in x;\ \chi(X, x, z),\ \forall a \in x\ (\chi(X, x, a) \Rightarrow a \in o') \vdash z \in o'$}
\LeftLabel{=-\textsc{subst}}
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\LeftLabel{$\wedge$-L}
\UnaryInfC{$z \in o,\ x \in X;\ \psi(X, x, z),\ \Sigma(X, o') \vdash z \in o'$}
\LeftLabel{$\exists$-L}
\RightLabel{(3)}
\UnaryInfC{$z \in o;\ \exists x \in X\ \psi(X, x, z),\ \Sigma(X, o') \vdash z \in o'$}
\LeftLabel{$\forall$-L}
\UnaryInfC{$z \in o;\ \forall a \in o\ \exists x \in X\ \psi(X, x, a),\ \Sigma(X, o') \vdash z \in o'$}
\LeftLabel{$\wedge$-L}
\UnaryInfC{$z \in o;\ \Sigma(X, o),\ \Sigma(X, o') \vdash z \in o'$}
\LeftLabel{$\subseteq$-R}
\RightLabel{(2)}
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\LeftLabel{$=_{\mathsf{Set}}$-R}
\dashedLine
\RightLabel{(1)}
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\end{prooftree}
$$

Proof idea for efficient extraction: compute an explicit definition by induction over the proof

Problem: what invariant?

Three standard restrictions

The adjectives

Cut-free, intuitionistic, focused

- All of the proofs we are going to be considering are cut-free
- We will ultimately drop the restriction to intuitionistic proofs...
- ...but ultimately enforce focusing anyway

Cut-freeness

$$\textsc{cut}\ \frac{\Gamma \vdash \phi, \Delta \qquad \Gamma, \phi \vdash \Delta}{\Gamma \vdash \Delta}$$

- Intuition: allows to introduce a lemma ϕ
- Other intuition: allows to *compose* proofs

Cut-elimination (Gentzen)

The cut rule does not allow to prove more sequents

- Effective argument, but cut-elimination is expensive

  (lower bound in $\mathcal{G}_3$ (Buss), i.e. above non-elementary)
- Related to computation in the λ-calculus (Curry-Howard)
- (Easier to define in the sequent calculus than in other systems)
- Cut-free proofs have a nice *subformula property*
- We will require cut-freeness essentially everywhere in the sequel

Intuitionism

At the intuitive level, reject the law of excluded middle/reasoning ad absurdum

$$\phi \vee \neg\phi \qquad\qquad \neg\neg\phi \implies \phi$$

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Conservativity for implicit definitions

If $\phi(i, o)$ is functional, then there is a formula $\chi(\vec{x})$ such that the conjoined formula

$$\phi^{\neg\neg}(i, o) \wedge \forall \vec{x}.\ \chi(\vec{x}) \vee \neg\chi(\vec{x})$$

can be proved to be functional in intuitionistic logic

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Actually non-trivial!! (I don't know a corresponding efficient algorithm)

Focusing

A normal form for proofs refining cut-freeness (Andreoli 90s)

Rough idea

Decompose proofs by forcing saturations by certain rules in *positive* and *negative* phase.

- Initially motivated by proof-search
- Like cut-elim, does not change provable statements
- To us: restricts the shape of proofs so much it allows to use simpler inductive invariants

  (probably a crutch, but we don't know how to work without it for now)

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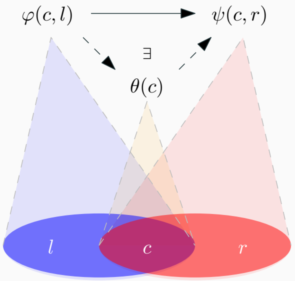
Complexity-wise (to the best of my knowledge)

A cut-free proof can be turned into a focused cut-free proof in exponential time.

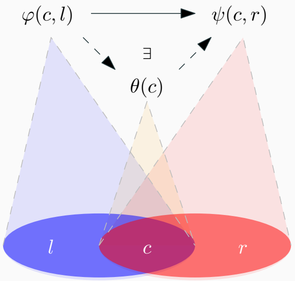


Craig interpolation

If $\varphi \Rightarrow \psi$, there exists θ such that

$$\varphi \Rightarrow \theta \qquad \text{and} \qquad \theta \Rightarrow \psi$$

Further, θ mentions *only* variables/relation symbols common to φ and ψ.



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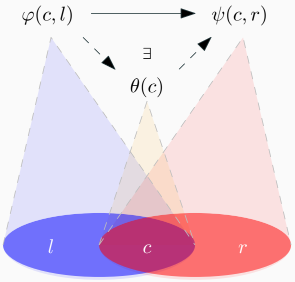
- Robust result

Δ_0-interpolation, intuitionistic/linear logic...

Preliminary: Craig interpolation



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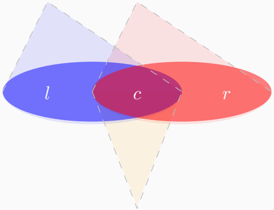
Δ_0-interpolation, intuitionistic/linear logic...

- θ linear-time computable from cut-free proofs
- Interpolation $\Rightarrow$ effective Beth definability

Our extraction procedure

$\Gamma(c, l),\ \Delta(c, r) \ \vdash \ l \subseteq r$



$\rightsquigarrow \exists\, e.\ \ l \subseteq e(c) \subseteq r$

Suppose $\Gamma(c, l), \Delta(c, r) \vdash \psi$.
Then we can compute $e(c)$ in NRC such that

Higher-type interpolation Lemma

- if ψ is $l = r$, then $\Gamma, \Delta \models l = e \wedge r = e$
- if ψ is $l \subseteq r$, then $\Gamma, \Delta \models l \subseteq e \wedge e \subseteq r$
- if ψ is $l \in r$, then $\Gamma, \Delta \models l \in e$

Stronger than standard interpolation

RHS depends on l

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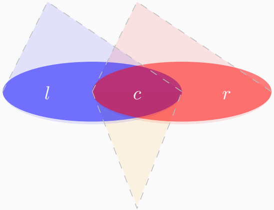

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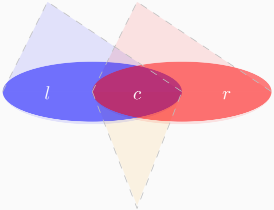

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Proof idea

Induction over the proof-tree; at some key steps

- Δ_0 interpolation
- NRC-definability of Δ_0-separation

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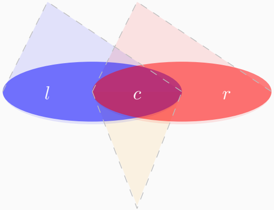

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Proof idea

Induction over the proof-tree; at some key steps

- Δ_0 interpolation
- NRC-definability of Δ_0-separation

Problem: does not generalize well to sequents with multiple conclusions

Handling classical proofs

New strategy: induction over the output type, some tedious proof theory and

(New and somewhat exciting!) NRC parameter collection theorem

Let L, R be sets of variables with $C = L \cap R$ and

- ϕ_L and $\lambda(z)$ Δ_0 formulas over L
- ϕ_R and $\rho(z, y)$ Δ_0 formulas over R
- r a variable of R and c a variable of C.

Suppose that we have a proof of $\phi_L \wedge \phi_R \;\Rightarrow\; \exists y \in_p r\, \forall z \in c.\;\; \lambda(z) \iff \rho(z, y)$

Then one may compute in polynomial time a NRC expression E with free variables in C such that

$$\phi_L \wedge \phi_R \quad\Longrightarrow\quad \{z \in c \mid \lambda(z)\} \in E$$

- By induction over *focused proofs*
- E is a *set* of candidate definitions for λ parameterized over the input

(reminiscent of a theorem of Chang and Makkai that yields definabilty from a proof of *fewness* rather than *uniqueness*)

More specifically

Lemma

Let L, R be sets of variables with $C = L \cap R$ and

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Then one may compute in polynomial time a NRC expression E and a Δ_0 θ over C s.t.

$$\phi_L \wedge \theta \implies \{z \in c \mid \lambda(z)\} \in E \qquad \text{and} \qquad \phi_R \vdash \theta$$

Intuitions:

- θ is an interpolant for a proof we are also computing on the fly
- Focusing allows to keep the invariant rather specific w.r.t. the r.h.s. formula

Key step: existential rule introducing the ``main'' formula

With $\mathcal{G} = \exists y \in_p r. \forall z \in c.\ \lambda(z) \iff \rho(z, y)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Theta_L, \Theta_R, x \in c \vdash \Delta_L, \Delta_R, \neg\rho(x, w), \lambda(x), \mathcal{G}}{\Theta_L, \Theta_R, x \in c \vdash \Delta_L, \Delta_R, \rho(x, w) \Rightarrow \lambda(x), \mathcal{G}}\vee \qquad \dfrac{\Theta_L, \Theta_R, x \in c \vdash \Delta_L, \Delta_R, \neg\lambda(x), \rho(x, w), \mathcal{G}}{\Theta_L, \Theta_R, x \in c \vdash \Delta_L, \Delta_R, \lambda(x) \Rightarrow \rho(x, w), \mathcal{G}}\vee}{\Theta_L, \Theta_R, x \in c \vdash \Delta_L, \Delta_R, \lambda(x) \Leftrightarrow \rho(x, w), \mathcal{G}}\wedge}{\Theta_L, \Theta_R \vdash \Delta_L, \Delta_R, \forall z \in c.\ (\lambda(z) \Leftrightarrow \rho(z, w)), \mathcal{G}}\forall}{\Theta_L, \Theta_R \vdash \Delta_L, \Delta_R, \mathcal{G}}\exists
$$

- Shape around the root of the tree guaranteed by focusing
- Applying the induction hypothesis we have

$$\Theta_L, x \in c \models \lambda(x), \Delta_L, \theta_1^{\mathsf{IH}} \vee \Lambda \in E_1^{\mathsf{IH}} \quad \text{and} \quad \Theta_L, x \in c \models \neg\lambda(x), \Delta_L, \theta_2^{\mathsf{IH}} \vee \Lambda \in E_2^{\mathsf{IH}}$$

$$\text{and} \quad \Theta_R \models \neg\rho(x, w), \Delta_R, \neg\theta_1^{\mathsf{IH}} \quad \text{and} \quad \Theta_R \models \rho(x, w), \Delta_R, \neg\theta_2^{\mathsf{IH}}$$

- So $\theta := \exists x \in c.\ \theta_1^{\mathsf{IH}} \wedge \theta_2^{\mathsf{IH}}$ and $E := \{\{x \in c \mid \theta_2^{\mathsf{IH}}\}\} \cup \bigcup \{E_1^{\mathsf{IH}} \cup E_2^{\mathsf{IH}} \mid x \in c\}$ works

Interpretations and multi-sorted definability

Interpretations

Nested collections can be regarded as multi-sorted structures

An object X of sort $\mathsf{Set}(\mathfrak{U} \times \mathsf{Set}(\mathfrak{U}))$

Sorts: $\mathfrak{U}, \mathsf{Set}(\mathfrak{U}), \mathfrak{U} \times \mathsf{Set}(\mathfrak{U})$

Function symbols: $\pi_1, \pi_2, \langle -, - \rangle$

Relation symbol: $\in_{\mathfrak{U}}$

Semantics: subobjects of X

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Interpretations: maps between finite structures defined by FO formulas

Can express

- product, disjoint union of structures $\mathfrak{M}, \mathfrak{N} \mapsto \mathfrak{M} \times \mathfrak{N}, \mathfrak{M} + \mathfrak{N}$
- definable substructures and quotients

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NRC and interpretations

For structures corresponding to nested collections,
NRC and Δ_0-interpretations coincide

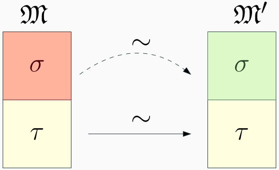
Remark: efficient translation from interpretations to NRC

Fix a theory Σ over two sorts τ and σ

Wlog: two sets of sorts

Multi-sorted implicit definability

σ is implicitly definable from τ when, for every $\mathfrak{M}, \mathfrak{M}' \models \Sigma$ and bijective homomorphism $\mathfrak{M}|_\tau \cong \mathfrak{M}'|_\tau$, there is a unique extension $\mathfrak{M} \cong \mathfrak{M}'$



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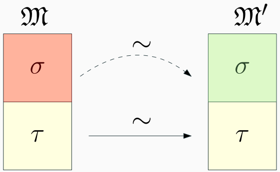

Reduction for implicit definition of nested transformations: single model where

- τ contains the input and $\mathfrak{U}$
- σ contains the output

possibly more complex than the input

From multi-sorted implicit definitions to explicit interpretations

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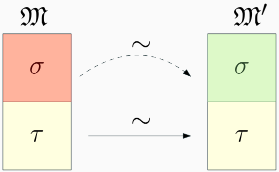

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Theorem

If σ is implicitly definable from τ, there is an interpretation of Σ into $\Sigma|_\tau$

Further perspectives

Natural question

Can we make the multi-sorted theorem effective?

- There is a natural notion of implicitly definable (although non-obvious)
- Effectivity is not an issue, but efficiency is
- (the intuitionistic case is easy)

Further topics

- Coq formalization with extraction
- Curry-Howard approach to the extraction of NRC terms

  ``untyped NRC'' treated by Sazonov
- Other settings for extraction from implicit definitions?