# Zigzag games, alternating infinite word automata and linear Monadic-Second order logic

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# Monadic Second-Order (MSO) logic and constructiveness

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- A fragment of Second-Order logic.
- Algorithmically decidable over

  $\mathbb{N}, \mathbb{Q}$, the infinite binary tree $\{0, 1\}^*$, ...
- Subsumes many verification logics. LTL, CTL, ...

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- Subsumes many verification logics. LTL, CTL, ...

## Decidable $\neq$ constructive

Soundness of decision procedures $\Longleftarrow$ non-constructive theorems.

- Over $\mathbb{N}$: infinite Ramsey theorem, weak König's Lemma.
- Over $\{0,1\}^*$: determinacy of infinite parity games.

## Syntax of MSO($\mathbb{N}$)

$$\varphi, \psi ::= n \in X \;\mid\; n < k \;\mid\; \exists n\, \varphi \;\mid\; \exists X\, \varphi \;\mid\; \neg\varphi \;\mid\; \varphi \wedge \psi$$

- Can be regarded as a subsystem of Second-Order Arithmetic
- Standard model: $n \in \mathbb{N}$, $X \in \mathcal{P}(\mathbb{N})$
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Corresponds exactly to sets recognizable by automata over infinite words.

- Infinite words: regard sets as sequences of bits through $\mathcal{P}(\mathbb{N}) \simeq 2^\omega$
- $\varphi(X_1, \dots X_k)$: formula over $\Sigma^\omega$ for $\Sigma = 2^k$

**Definition**

A non-deterministic Büchi automaton (NBA) $\mathcal{A} : \Sigma$ is a tuple $(Q, q_0, \delta, F)$

- $Q$ is a finite set of states, $q_0 \in Q$
- transition function $\delta : \Sigma \times Q \to \mathcal{P}(Q)$
- $F \subseteq Q$ *accepting states*

Recognizes languages of infinite words $\mathcal{L}(\mathcal{A}) \subseteq \Sigma^\omega$:

$$w \in \mathcal{L}(\mathcal{A}) \text{ iff there is a run over } w \in \Sigma^\omega \text{ hitting } F \text{ infinitely often}$$

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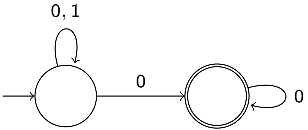
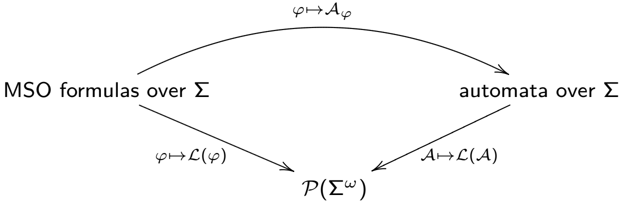
Example:




$\mathcal{L}(\mathcal{A}) =$ streams with finitely many 1.

# MSO/automata correspondance



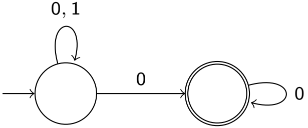

**Decidability [Büchi (1962)]**

MSO over infinite words is decidable.

- Proof idea: automata theoretic-construction for each logical connective.
- Hard case for infinite words: negation $\neg$.

corresponds to *complementation*

For finite word automata: easy complementation for *deterministic* automata.

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# Complementation, determinization and constructivity

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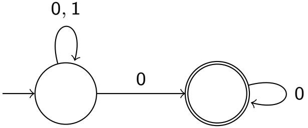


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Non-deterministic Büchi automata can be determinized into *Rabin automata*.

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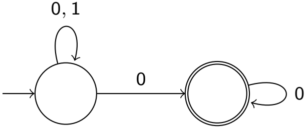


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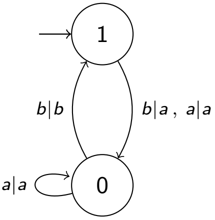
- Büchi's original complementation procedure: w/o determinization.
- Effective algorithms for automata . . .
- . . . but non-constructive proofs of soundness!

usual proofs: infinite Ramsey theorem, weak König's lemma

# Church's synthesis (1/2): causal functions




**Causal/synchronous stream functions $f : \Sigma^\omega \to \Gamma^\omega$**

- Interpret $n \in \mathbb{N}$ as **time steps**.
- Lifted from functions $\hat{f} : \Sigma^+ \to \Gamma$ as

$$\begin{array}{cccc} \hat{f} : & \Sigma^\omega & \to & \Gamma^\omega \\ & s & \mapsto & n \mapsto f(s(0)\dots s(n)) \end{array}$$

  i.e., the output does not depend on the future.
- Focus on *finite-state* causal functions.
  (Correspond to *Mealy machines*)

- All f.s. causal functions are recursive.
- All causal functions are continuous.
- Some recursive functions are not causal. $w \longmapsto n \mapsto w_{n+1}$

# Church's synthesis (2/2): the Büchi-Landweber theorem

## Church's synthesis problem

Given a formula $\varphi(X, Y)$, find a f. s. causal $f : \Sigma^\omega \to \Gamma^\omega$ such that

$$\forall w \; \varphi(w, f(w))$$

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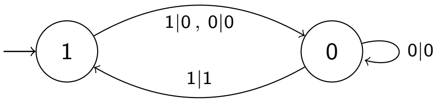
Example (inspired from [Thomas (2008)]):

- $\varphi(X, Y) \quad \equiv \quad (X \text{ infinite} \Rightarrow Y \text{ infinite}) \quad \text{and} \quad \forall i \; (i \in Y \Rightarrow i + 1 \notin Y)$




## Theorem [Büchi-Landweber (1969)]

Algorithmic solution for $\varphi(X, Y)$ in MSO.

- Algorithmically costly. . .

# MSO and proofs

MSO can also be seen as a classical axiomatic theory

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Church's synthesis reminiscent of extraction from proofs:

$$\text{MSO} \vdash \forall x \exists y \, \varphi(x, y) \qquad \overset{?}{\Longrightarrow} \qquad \exists f \text{ f.s. causal} \quad \forall x \, \varphi(x, f(x))$$

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**Classical theorems in MSO**

- Excluded middle (subtle point $\{0, 1\}^\omega$ vs $\mathcal{P}(\mathbb{N})$)
- The infinite pigeonhole principle
- Instances of additive Ramsey

$\leadsto$ No algorithmic witnesses for $\forall\exists$ theorems.

# Extraction from proofs

**Goal**: a refinement of MSO($\mathbb{N}$) with extraction for **causal** functions.

- Toward semi-automatic approach to synthesis.
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## Analogous example: extraction for intuitionistic arithmetic (HA)

If $HA \vdash \forall x \exists y \varphi(x, y)$, there is an algorithm computing

$$f : \mathbb{N} \to \mathbb{N} \text{ recursive} \qquad \text{such that} \qquad \forall x \quad \varphi(x, f(x))$$

- A subset of classical arithmetic (PA).
- As expressive as classical arithmetic. ($\varphi \mapsto \varphi^{\neg\neg}$)
- Can be refined to System T functions. [Gödel (1930s)]

## Analogy

| Classical system | MSO($\mathbb{N}$) | PA |
|---|---|---|
| Realizers | Causal functions | System T |
| Intuitionistic system | ??? | HA |

# Synchronous MSO (SMSO) [P., Riba (2017)]

Intuitionistic version of MSO

$$\varphi, \psi \quad ::= \quad \alpha \quad | \quad \varphi \wedge \psi \quad | \quad \exists X \ \varphi \quad | \quad \neg\varphi$$

Quantification over individuals encoded as usual

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MSO $\vdash \varphi$ if and only if SMSO $\vdash \neg\neg\varphi$

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SMSO $\vdash \exists y \; \neg\neg\varphi(x, y)$ iff there is a f.s. causal $f$ s.t. MSO $\vdash \forall x \; \varphi(x, f(x))$

- Proofs $\varphi \vdash \psi$ interpreted as simulations between ND automata.

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Polarity restriction

# A linear refinement LMSO [P., Riba (2018)]

- Polarized system with dualities.
- Requires the introduction of **linear** connectives.

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$$\varphi, \psi \quad ::= \quad \alpha \mid \varphi \otimes \psi \mid \varphi \parr \psi \mid \varphi \multimap \psi \mid \forall X \varphi \mid \exists X \varphi \mid !\varphi^- \mid ?\varphi^+ \mid \dots$$

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Alternating $(\forall, \exists, \otimes, ⅋, \multimap)$

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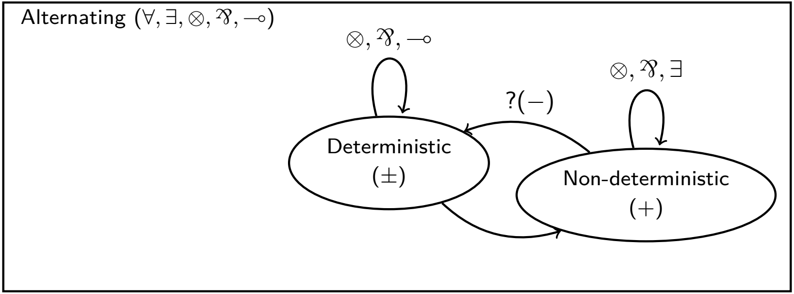


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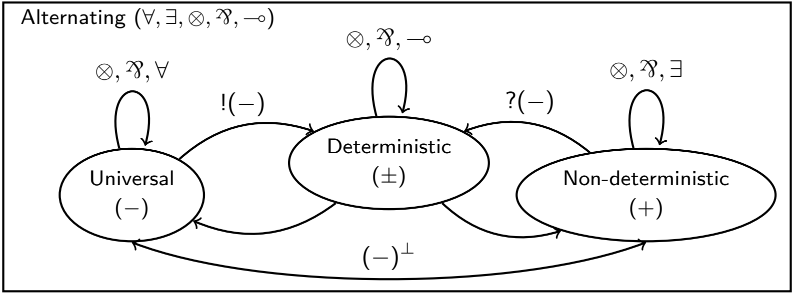


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# Expressivity and proof extraction for LMSO

**Conservativity**

$$\begin{array}{ccc} \text{LMSO} & \to & \text{MSO} \\ \varphi & \mapsto & \lceil \varphi \rceil \end{array}$$

If LMSO $\vdash \varphi$, then MSO $\vdash \lceil \varphi \rceil$.

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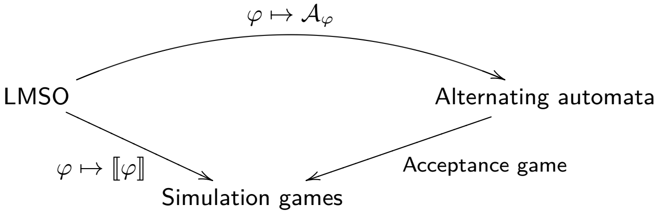

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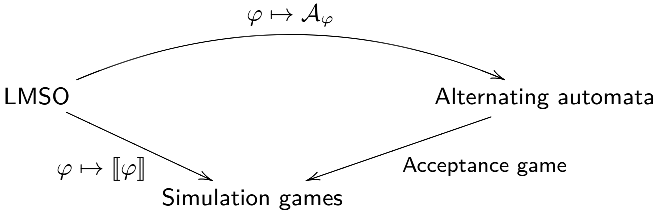


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For every $\varphi$, there is a realizer $\quad (\varphi \multimap \bot) \multimap \bot \quad \multimap \quad \varphi$

*but no canonical iso in general!*

- Also holds in DC if the base satisfies choice.

# Why automata?

The above logic can be defined without reference to automata.

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$\rightsquigarrow$ A purely logical reformulation of LMSO using categorical semantics.

**Goals**

- Purely syntactic transformations.
- Understand links with typed realizability and Dialectica.

# Finite-state causal functions as terms

Define the category $\mathbb{M}$ of causal functions

- Objects: sets of streams $\Sigma^\omega$ for $\Sigma$ finite
- Morphisms: finite-state causal functions
- Cartesian products $\Sigma^\omega \times \Gamma^\omega \simeq (\Sigma \times \Gamma)^\omega$, but **not** cartesian-closed

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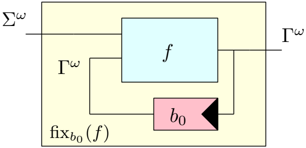
## Inductive presentation

$$\frac{f : \Sigma \to \Gamma}{f^\omega : \Sigma^\omega \to \Gamma^\omega} \qquad \frac{f : \Sigma^\omega \times \Gamma^\omega \to \Gamma^\omega \quad b_0 \in \Gamma}{\mathrm{fix}_{b_0}(f) : \Sigma^\omega \to \Gamma^\omega}$$

+ closure under composition




$\approx$ guarded recursion $\mathrm{fix} : A^{\blacktriangleright A} \to A$

topos of trees

## MSO($\mathbb{N}$) as an equational logic over $\mathbb{M}$

### FOM (First-Order Mealy)

$$\varphi, \psi \quad ::= \quad \mathrm{t} =_{\Sigma^\omega} \mathrm{u} \;\mid\; \varphi \wedge \psi \;\mid\; \neg\varphi \;\mid\; \exists x \in \Sigma^\omega\ \varphi$$

- Typed variables stand for streams, terms for every f.s. causal functions.

### Proposition

FOM and MSO($\mathbb{N}$) are interpretable in one another.

- Justifies focusing on FOM.

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## Tarskian semantics (categorical logic)

- Regard $\mathbb{M}$ as a multi-sorted Lawvere theory.

$\rightsquigarrow$ Tarskian semantics $\approx$ indexed category, from global section functor $\mathbf{\Gamma}$

$$\begin{aligned} \mathbf{\Gamma} : \quad \Sigma^\omega \quad &\longmapsto \quad \mathrm{Hom}_{\mathbb{M}}\left(1^\omega, \Sigma^\omega\right) \\ \Sigma^\omega \quad &\longmapsto \quad \left(\mathcal{P}\left(\mathbf{\Gamma}\left(\Sigma^\omega\right)\right), \subseteq\right) \end{aligned}$$

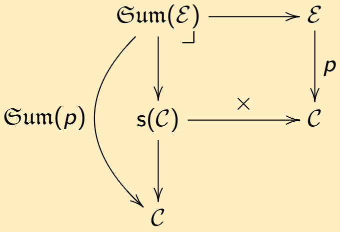
# SMSO and the simple fibration

Simple slice $\mathcal{C}//X$ = full subcategory of $\mathcal{C}/X$ with objects

$$X \times Y \xrightarrow{\pi} X$$

$\rightsquigarrow$ the simple fibration $\mathsf{s}(\mathcal{C}) \to \mathcal{C}$

## The construction $\mathfrak{Sum}$




- $\mathfrak{Sum}(p)$-predicate: $(U, \varphi(a, u))$

  $U$ object of $\mathcal{C}$, $\varphi$ over $A \times U$ (in $p$)

  $\approx \exists u : U\ \varphi(a, u)$

- Freely adds existential quantifications

  (simple sums)

- Reminiscent of typed realizability

  realizers in $\mathcal{C}$

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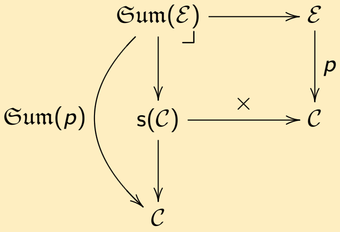


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## Reconstructing SMSO

Simulations of non-determinstic automata $\approx$ $\mathfrak{Sum}$ applied to FOM

# Linking LMSO with Dialectica

## Fibered Dialectica [Hyland (2001)]

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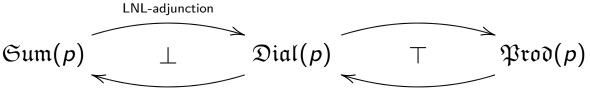

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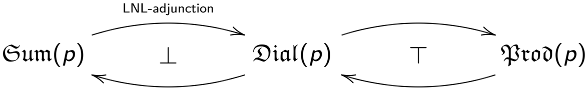

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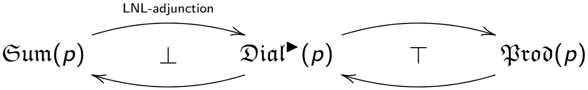


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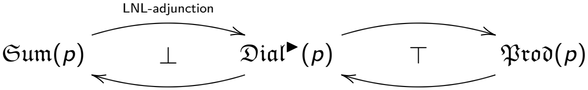


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LNL-adjunction

$$\mathfrak{Sum}(p) \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathfrak{Dial}^{\blacktriangleright}(p) \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathfrak{Prod}(p)$$

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- Polarity restrictions $\approx$ model of LMSO (restricted exponentials)

# In a nutshell

## Summary

- Realizability models based on simulations between automata
- Abstract reformulation — link with Dialectica and typed realizability
- Complete extension of LMSO — omitted from the talk [P., Riba (2019)]

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## Related work

- Fibrations of tree automata — [Riba (2015)]
- Good-for-games automata
  [Henziger, Piterman (2006), Kuperberg Skrzypczak (2015)]

# Final word

## Some further questions

- Realizability for *continuous* functions $\Sigma^\omega \to \Gamma^\omega$?
- Extensions of $\mathfrak{Dial}^{\blacktriangleright}$ for fibrations over the topos of trees?

  $\mathfrak{Fam}(\mathfrak{Fam}(p^{\mathrm{op}})^{\mathrm{op}})$ instead of $\mathfrak{Dial}(p)$
- Undecidability of the equational logic of higher-order extensions of FOM?
- Reconstructing zizgag games as the final coalgebra for $\mathcal{C} \mapsto \mathcal{C}_{\oplus\&}$?

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Thanks for your attention! Questions?

# Induction and comprehension

$\mathsf{RCA}_0$ is defined by restricting *induction* and *comprehension*

## Comprehension axiom

For every formula $\phi(n)$ (with $X \notin FV(\phi)$)

$$\exists X\ \forall n \in \mathbb{N}\ \ (\phi(n) \Leftrightarrow n \in X)$$

- $\mathsf{RCA}_0$: restricted to $\Delta^0_1$ formulas

recursive comprehension

## Induction axiom

To prove that $\forall n \in \mathbb{N} \phi(n)$ it suffices to show

- $\phi(0)$ holds
- for every $n \in \mathbb{N}$, $\phi(n)$ implies $\phi(n+1)$

- $\mathsf{RCA}_0$: restricted to $\Sigma^0_1$ formulas.

$\exists n\ \delta(n)$ with $\delta \in \Delta^0_1$

- Equivalent to minimization principles and comprehension for finite sets.

# Additive Ramsey over $\omega$

For any linear order $(P, <)$ write $[P]^2$ for $\{(i,j) \in P^2 \mid i < j\}$ and fix a finite monoid $(M, \cdot, e)$.
Call $f : [P]^2 \to M$ *additive* when $f(i,j) \cdot f(j,k) \ = \ f(i,k)$ for all $i < j < k$

## Additive Ramsey

For any additive $f : [P]^2 \to M$, there is an unbounded monochromatic $X \subseteq P$ (s.t. $|f([X]^2)| = 1$).

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## Additive Ramsey

For any additive $f:[P]^2 \to M$, there is an unbounded monochromatic $X \subseteq P$ (s.t. $|f([X]^2)| = 1$).

## Theorem

Over $RCA_0$, additive Ramsey over $\omega$ is equivalent to $\Sigma^0_2$-induction.

Direct proof: "as usual" for additive Ramsey (factored through an ordered variant in the paper)

## $\Pi^0_2$-induction from additive Ramsey

Consider equivalently comprehension for sets bounded by $n$ for $\exists^\infty k \;\; \delta(x,k)$.
Define the coloring $f:[\omega]^2 \to 2^n$ as $\quad f(i,j)_x = \max_{i \le l < j} \delta(x,l)$.

Apply additive Ramsey and consider the color $X$ of the monochromatic set; we have

$$x \in X \qquad \Leftrightarrow \qquad \exists^\infty \delta(x,k)$$

# Combinatorics for coloring over $\mathbb{Q}$

Let $D$ be a dense linear order ($\simeq \mathbb{Q}$).
A function $f : D \to X$ is called *homogeneous* if $f^{-1}(x)$ is either dense or empty for every $x \in X$.

## The shuffle principle

For any coloring $c : \mathbb{Q} \to [\![0, n]\!]$, there is $I \subseteq_{\mathrm{conv}} Q$ such that $c\big|_I$ is a shuffle.

- the key additional principle behind the usual inductive argument in [Carton, Colcombet, Puppis (2015)]

## Shelah's additive Ramseyan theorem

Let $M$ be a monoid. For every map $f : [\mathbb{Q}]^2 \to M$ such that $f(q,r)f(r,s) = f(q,s)$, there exists an interval $I \subseteq \mathbb{Q}$ and a finite partition into finitely many dense sets $D_i$ of $I$ such that $f$ is constant over each $[D_i]^2$.

- the key additional principle behind the usual inductive argument in [Shelah (1975)]

# The Büchi-Landweber theorem

Consider a formula $\varphi(u, x)$. $(u \in U^\omega, x \in X^\omega)$

$\rightsquigarrow$ Infinite 2-player game $\mathcal{G}_\varphi$ between **P** and **O**.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **O** | $x_0$ | | $x_1$ | | ... | $x_n$ | | ... |
| **P** | | $u_0$ | | $u_1$ | | | $u_n$ | |

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causal functions

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### Theorem [Büchi-Landweber (1969)]

Suppose $\varphi$ is MSO-definable. The game $\mathcal{G}_\varphi$ is determined:

- Either there exists a finite-state **P**-strategy $s_{\mathbf{P}}(x)$ s.t. $\forall x \in X^\omega \quad \varphi(s_{\mathbf{P}}(x), x)$
- Or there exists a finite-state **O**-strategy $s_{\mathbf{O}}(u)$ s.t. $\forall u \in U^\omega \ \neg\varphi(u, s_{\mathbf{O}}(u))$

# The realizability notion for SMSO

## Uniform non-deterministic automata

Tuples $\mathcal{A} = (Q, q_0, U, \delta_{\mathcal{A}}, \Omega_{\mathcal{A}}) : \Sigma$ where

- $U$ a set of *moves* — $\simeq$ amount of non-determinism
- transition function $\delta_{\mathcal{A}} : \Sigma \times Q \times U \to Q$ — induces $\delta^*_{\mathcal{A}} : \Sigma^\omega \times U^\omega \to Q^\omega$
- $\Omega_{\mathcal{A}} \subseteq Q^\omega$ reasonable acceptance condition — (parity, Muller, ...)

- Same definable languages $\mathcal{L}(\mathcal{A}) = \{w \mid \exists u\ \delta^*_{\mathcal{A}}(w, u)\}$ — $U \simeq Q$

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## Simulations $\mathcal{A} \Vdash f : \mathcal{B}$

Finite-state causal function $f : \Sigma^\omega \times U^\omega \to V^\omega$ such that

$$\forall w \in \Sigma^\omega \forall u \in U^\omega \qquad \delta^*_{\mathcal{A}}(w, u) \in \Omega_{\mathcal{A}} \quad \Rightarrow \quad \delta^*_{\mathcal{A}}(w, f(w, u)) \in \Omega_{\mathcal{B}}$$

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- If $\mathcal{A} \Vdash \mathcal{B}$, then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$
- Natural interpretation for $\exists$, $\wedge$ and $\neg$ for deterministic automata...

# Alternating uniform automata

Define a notion of *alternating* uniform automata $(Q, q_0, U, X, \delta, \Omega)$

- sets of **P**-moves $U$ and **O**-moves $X$
- $\delta : \Sigma \times Q \times U \times X \to Q$
- $w \in \mathcal{L}(\mathcal{A})$ iff **P** wins an *acceptance game*

## Simulation game

| | $(U\ ,\ X)$ | $\multimap$ | $(V,\ Y)$ |
|---|---|---|---|
| | | $\vdots$ | |
| **O** | $u_n$ | | |
| **P** | | | $v_n$ |
| **O** | | | $y_n$ |
| **P** | $x_n$ | | |
| | | $\vdots$ | |

**P** wins iff

$\langle u, x\rangle$ **P**-winning $\Rightarrow$ $\langle v, y\rangle$ **P**-winning

- $X \simeq 1 \quad \rightsquigarrow$ non-deterministic uniform automata
- $U \simeq X \simeq 1 \quad \rightsquigarrow$ deterministic automata

trivial simulations