Comparison-free polyregular functions

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University of Oxford, United Kingdom

ICALP 2021 --- track B

Introduction: string-to-string transducers

Topic of this talk: certain basic computation models for string-to-string functions

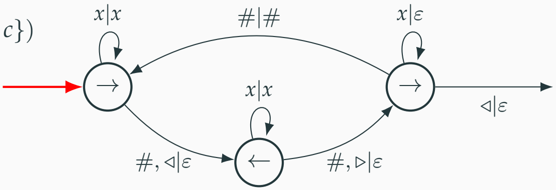
Two-way transducers: executive summary

Finite set of states + bidirectional reading head + output produced from left to right

Example:

$$\texttt{mapPalin} : \{a,b,c,\#\}^* \longrightarrow \{a,b,c,\#\}^*$$
$$w_1\#\ldots\#w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\#\ldots\#w_n \cdot \texttt{reverse}(w_n)$$

$(x \in \{a,b,c\})$



Output:

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	a	$\triangleleft$

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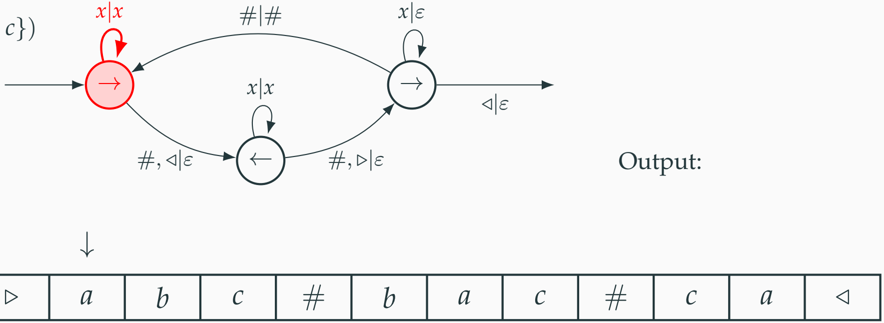

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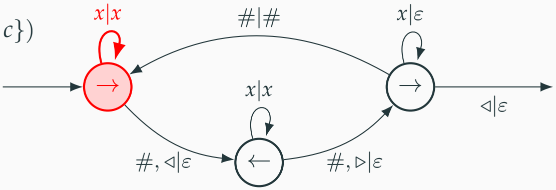
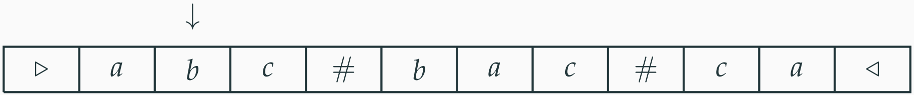


Output:
a



Deterministic two-way transducers (2DFT)

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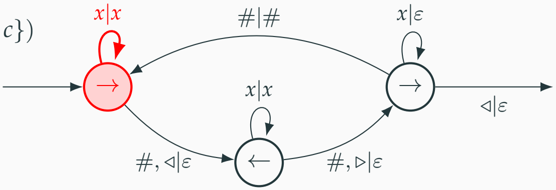
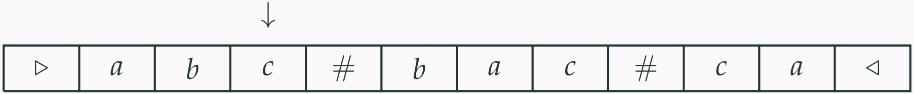


Output:
ab



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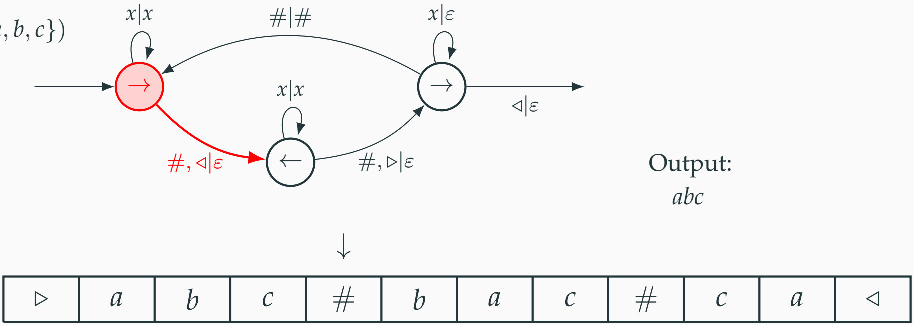

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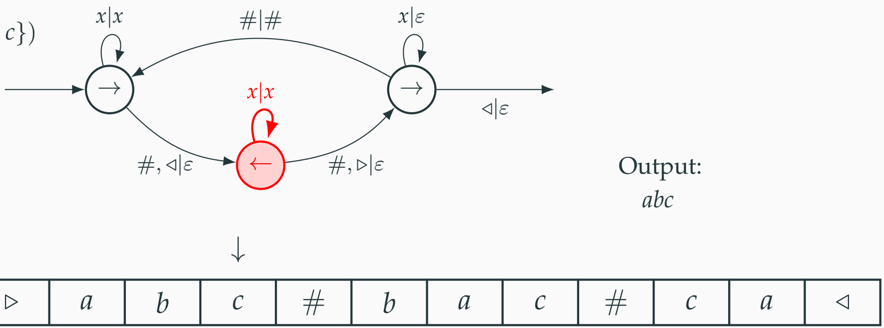

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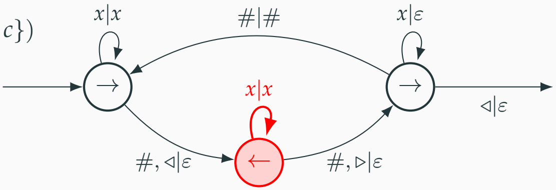
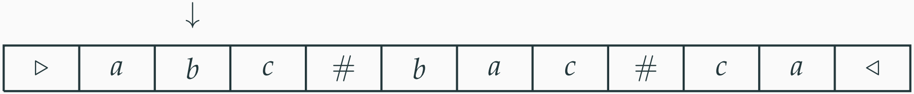


Output:
abcc



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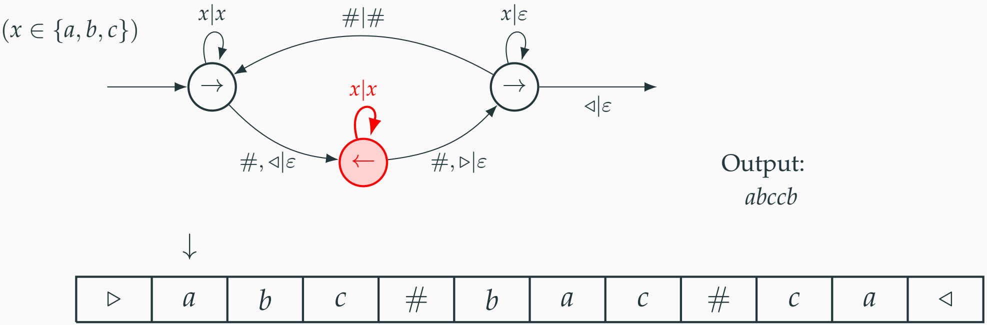

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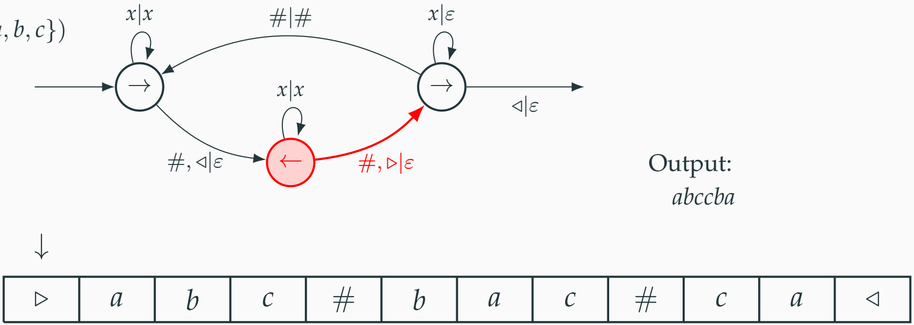

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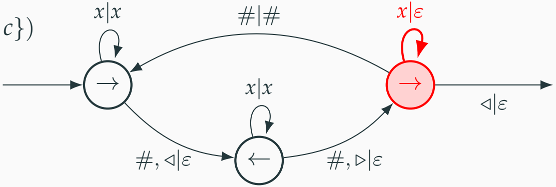
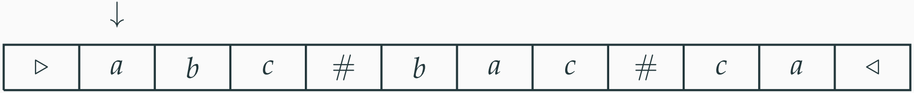


Output:
abccba



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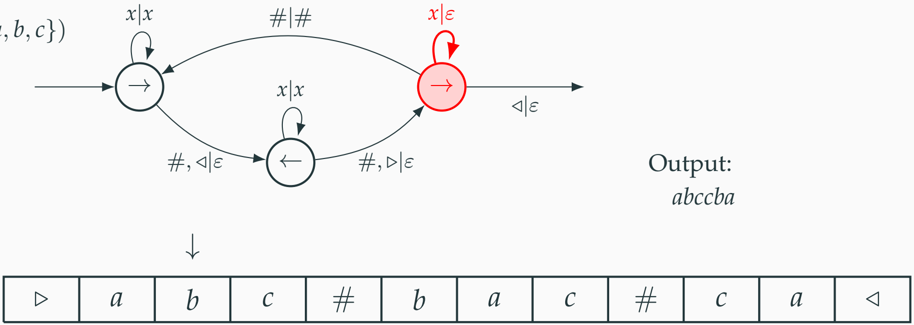

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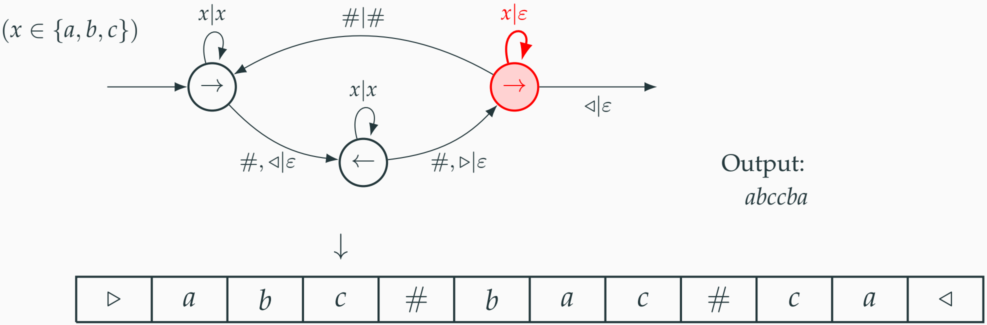

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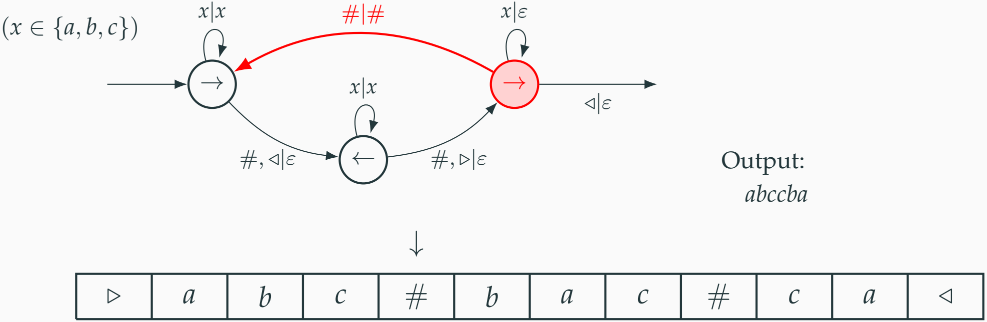

Output:
abccba

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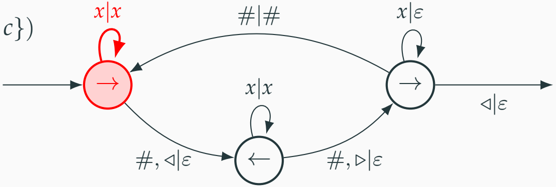
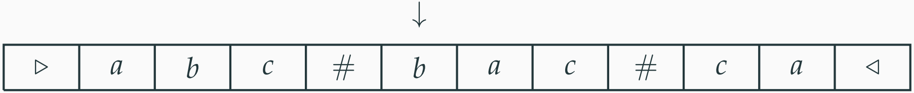


Output:
abccba#



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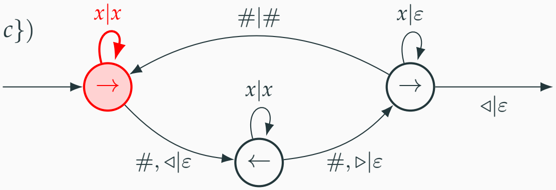
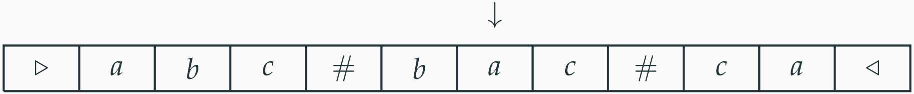


Output:
abccba#b



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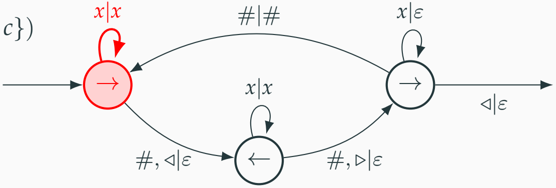
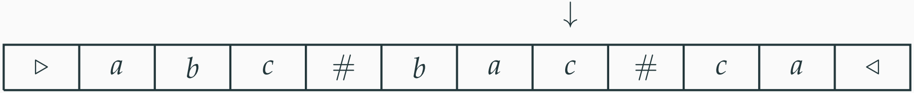


Output:
$abccba\#ba$



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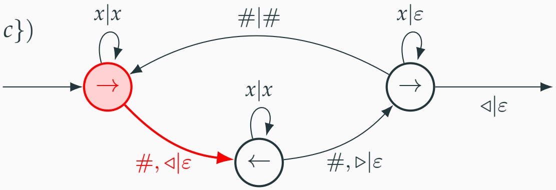
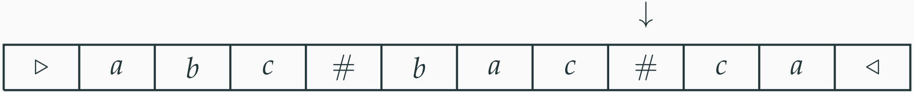


Output:

abccba#bac



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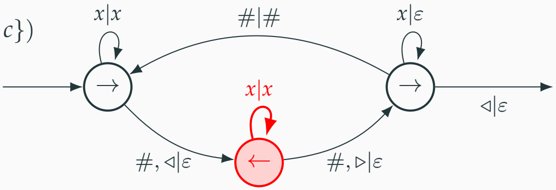

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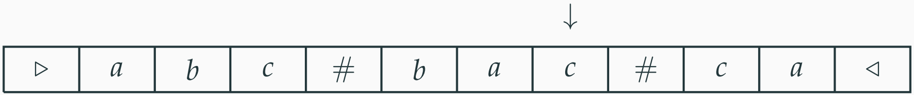

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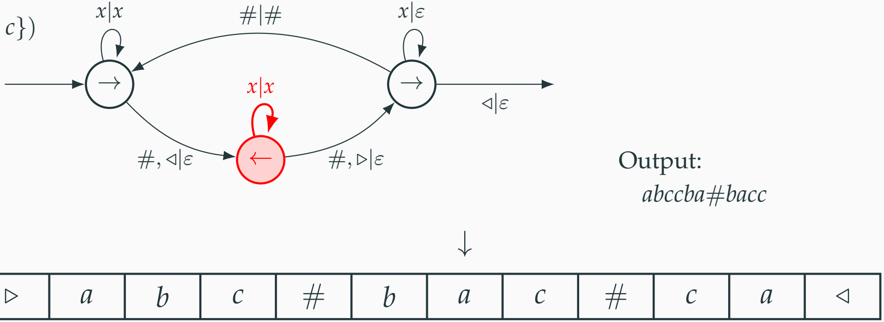

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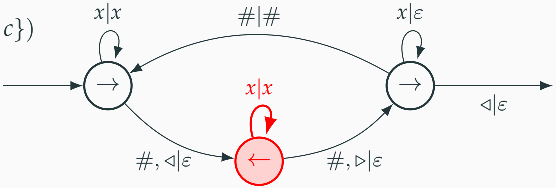
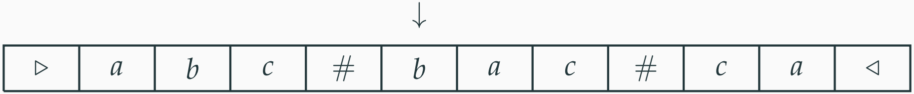


Output:

abccba#bacca



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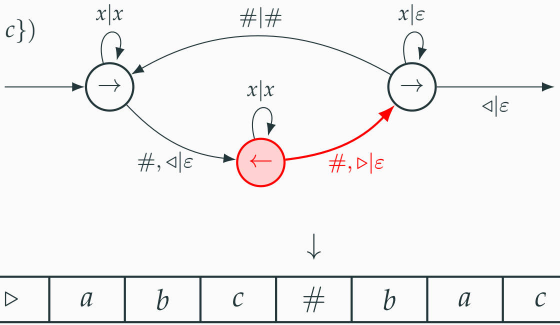

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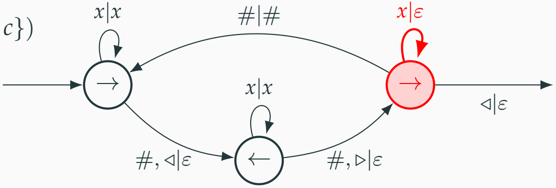

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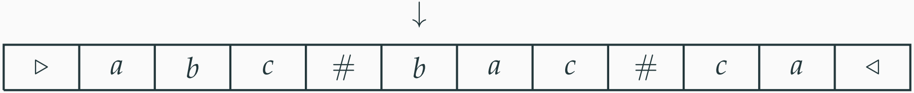

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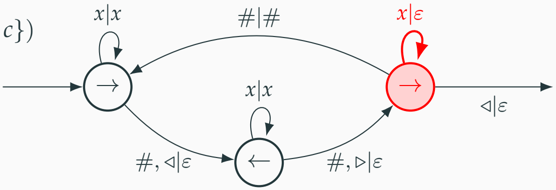

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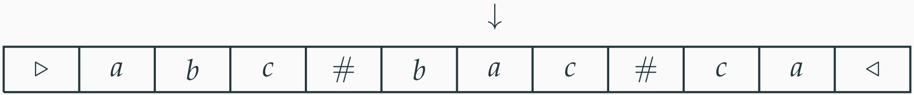

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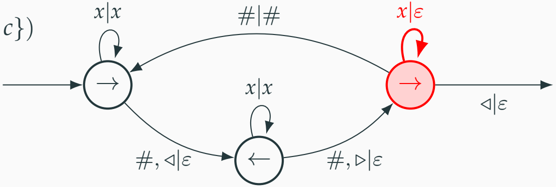

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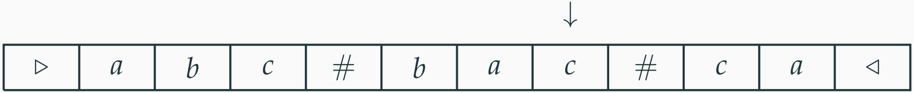

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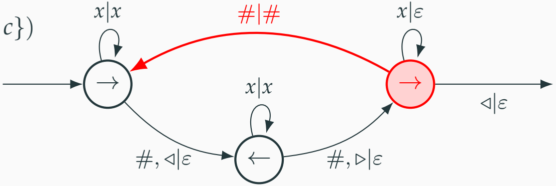

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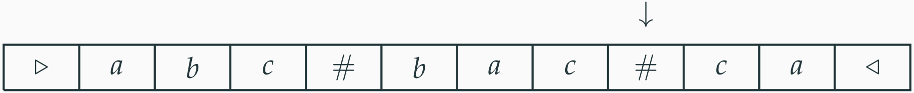
abccba#baccab



Deterministic two-way transducers (2DFT)

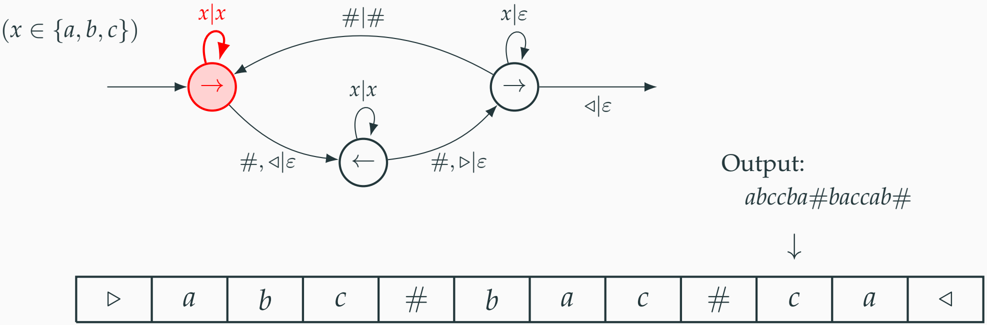
Topic of this talk: certain basic computation models for string-to-string functions

Two-way transducers: executive summary

Finite set of states + bidirectional reading head + output produced from left to right

Example:

$$\begin{array}{rccl} \texttt{mapPalin}: & \{a,b,c,\#\}^* & \longrightarrow & \{a,b,c,\#\}^* \\ & w_1\#\ldots\#w_n & \longmapsto & w_1 \cdot \texttt{reverse}(w_1)\#\ldots\#w_n \cdot \texttt{reverse}(w_n) \end{array}$$



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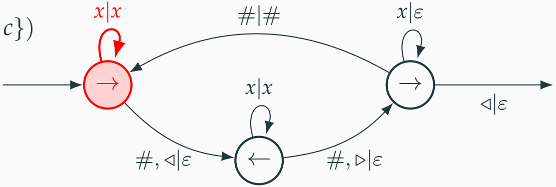

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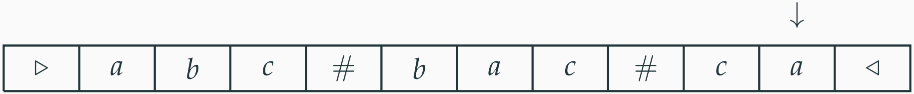
abccba#baccab#c



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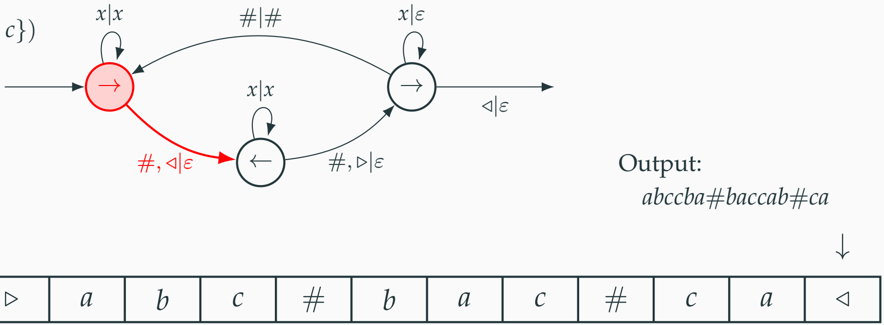

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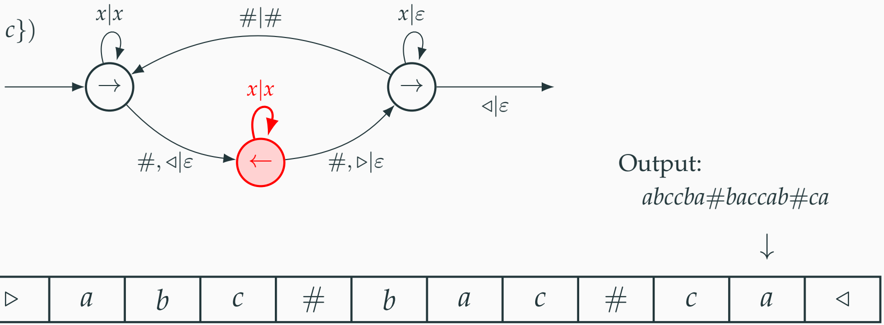

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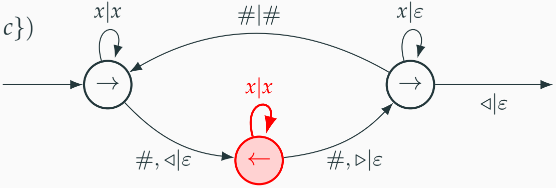
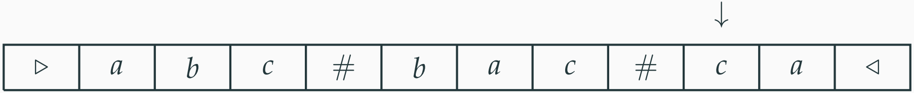


Output:

abccba#baccab#caa



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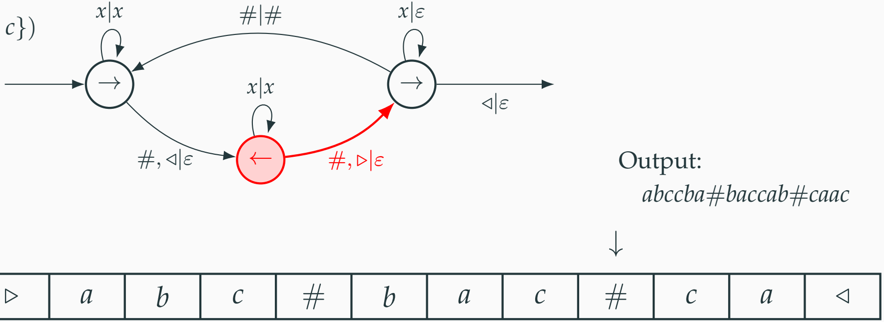

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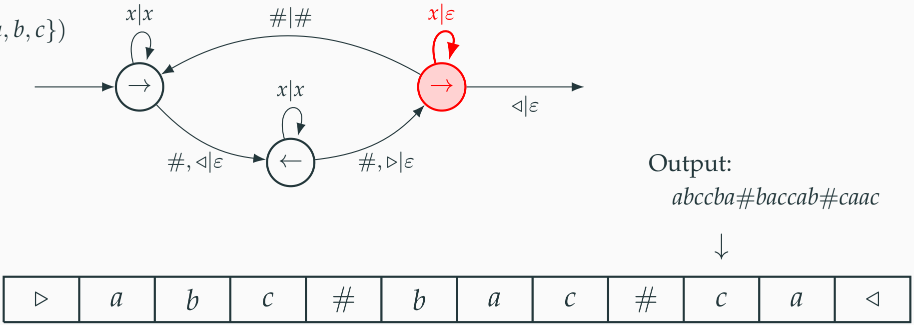

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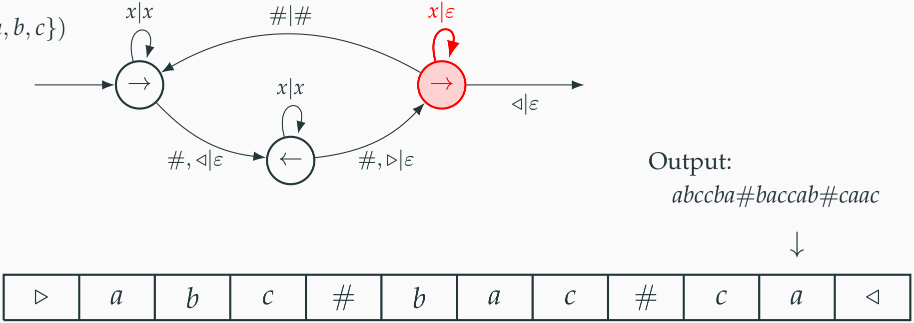

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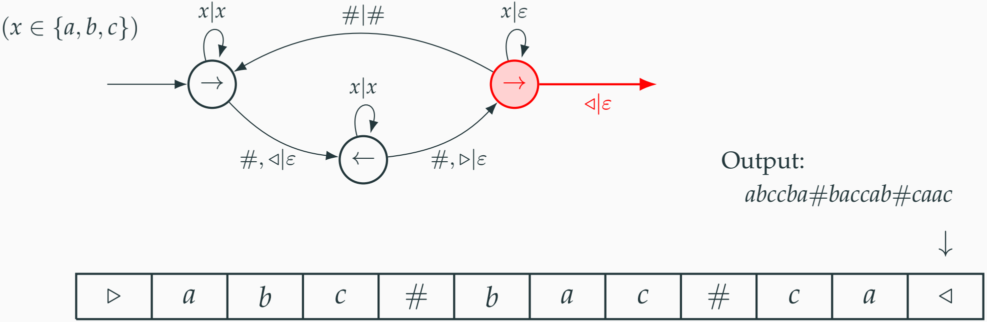

Regular functions

Functions $\Sigma^* \to \Gamma^*$ definable by 2DFTs are called **regular functions**

Properties of regular functions

- Linear growth: $|f(w)| = O(|w|)$
- Closed under composition (if $f : \Gamma^* \to \Sigma^*$ and $g : \Sigma^* \to \Pi^*$ are regular then so is $g \circ f$)
- L regular $\implies f^{-1}(L)$ regular

Alternative characterizations

- Via Monadic Second-Order logic (MSO transductions)
- Copyless streaming string transducers
- Various functional programming or regexp-like (declarative) formalisms
- (recent work of ours) Minimal linear λ-calculus and Church encodings [Nguyễn,Noûs,P. 2020]

Polyregular functions

Polyregular functions:

- A larger class of string-to-string transductions
- Garnered significant attention recently, starting with [Bojańczyk 2018]

Properties

- *Polynomial* growth: $|f(w)| = O(|w|^k)$
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- Closed under composition

Characterizations [Bojańczyk 2018; Bojańczyk, Kiefer & Lhote 2019]

- Multidimensional MSO interpretations
- Imperative nested loop programs
- Simply typed λ-calculus augmented with a list type and some list manipulation primitives
- Composition closure of [regular functions $\cup$ "squaring with underlining"]

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- ***k*-pebble string-to-string transducers**

k-pebble transducers: executive summary

Finite set of states + a stack of two-way reading heads of height $\leq k$

- Heads can be moved, pushed, popped
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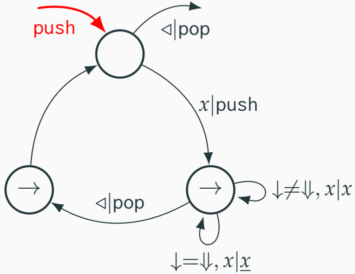
- 1-pebble transducers $\cong$ 2DFTs

Example: "squaring with underlining" ($k = 2$)



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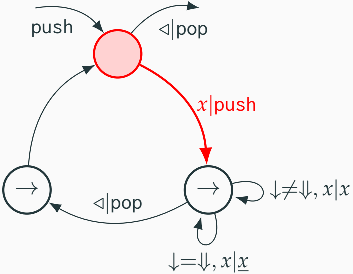

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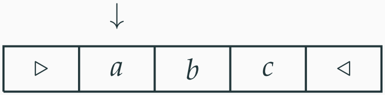

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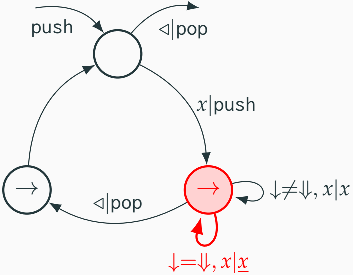

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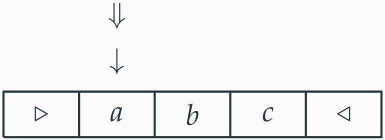

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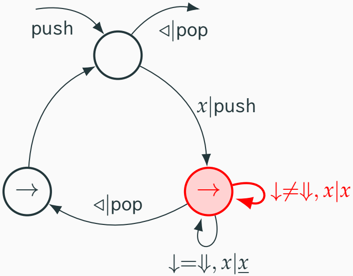

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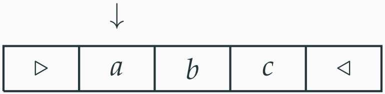

output = $\underline{a}$

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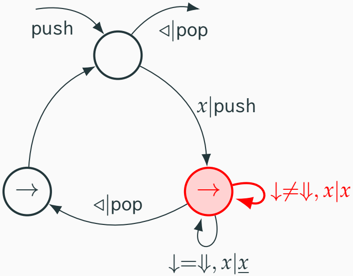

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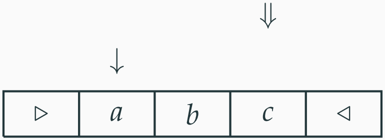

output $= \underline{a}b$

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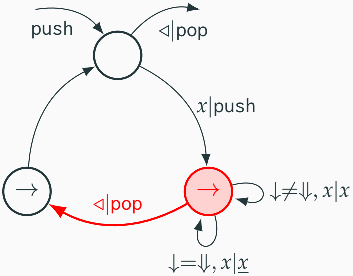

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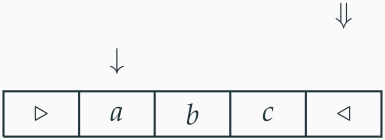

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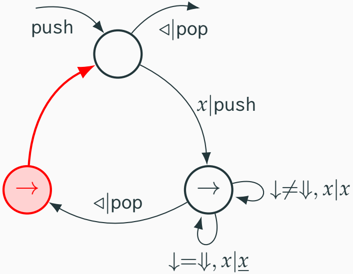

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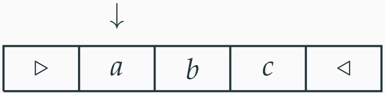

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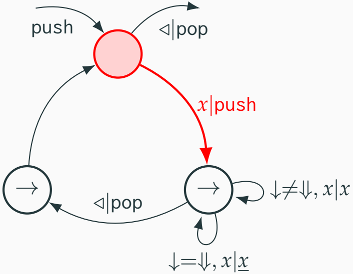

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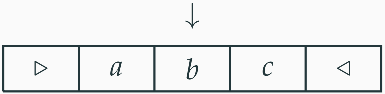

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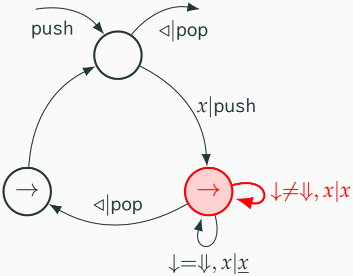

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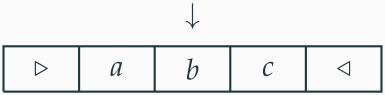

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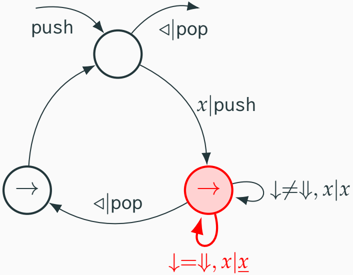

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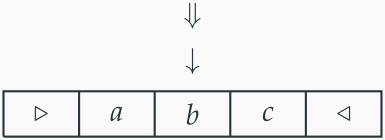

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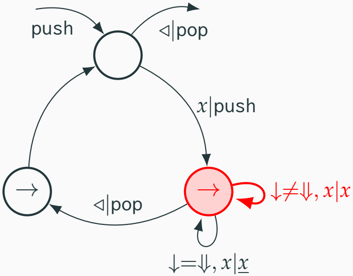

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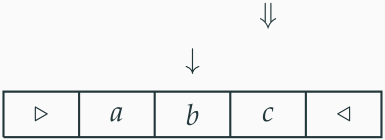

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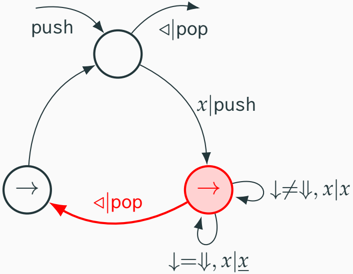

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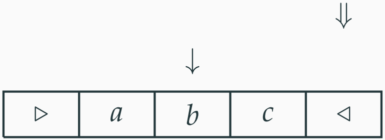

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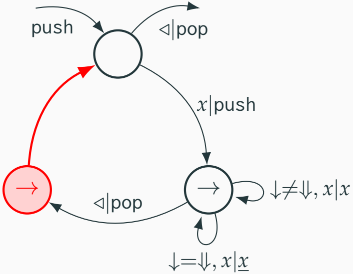

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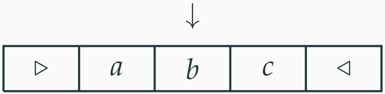

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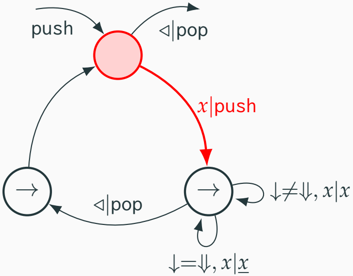

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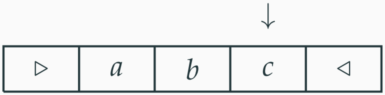


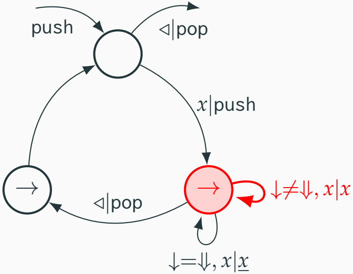
output = $\underline{a}bca\underline{b}c$

k-pebble transducers: executive summary

Finite set of states + a stack of two-way reading heads of height $\leq k$

- Heads can be moved, pushed, popped
- Arbitrary comparisons between heads in the stack
- 1-pebble transducers $\cong$ 2DFTs

Example: "squaring with underlining" ($k = 2$)



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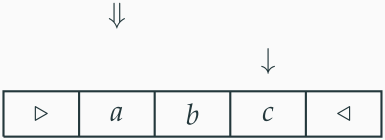

output $= \underline{a}bca\underline{b}c$

k-pebble transducers: executive summary

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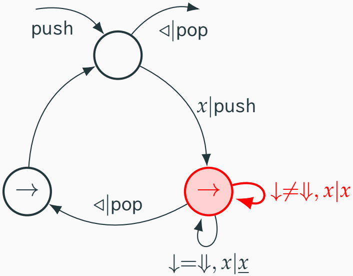
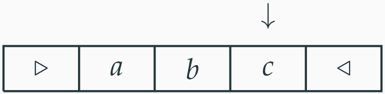


$$\begin{array}{rccc} \texttt{squaring}: & \Sigma^* & \to & (\Sigma \cup \underline{\Sigma})^* \\ & aab & \mapsto & \underline{a}aba\underline{a}baa\underline{b} \end{array}$$

$\Downarrow$



output $= \underline{a}bca\underline{b}ca$

Pebble transducers

k-pebble transducers: executive summary

Finite set of states + a stack of two-way reading heads of height $\leq k$

- Heads can be moved, pushed, popped
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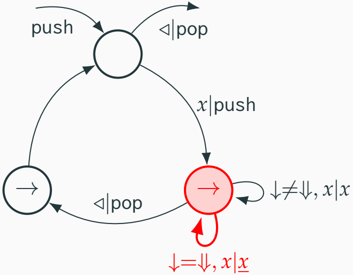
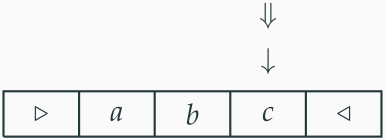
Example: "squaring with underlining" ($k = 2$)



$$\texttt{squaring}: \quad \Sigma^* \to (\Sigma \cup \underline{\Sigma})^*$$
$$aab \mapsto \underline{a}aba\underline{a}baa\underline{b}$$



output $= \underline{a}bca\underline{b}cab$

k-pebble transducers: executive summary

Finite set of states + a stack of two-way reading heads of height $\leq k$

- Heads can be moved, pushed, popped
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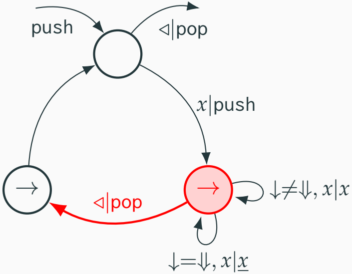
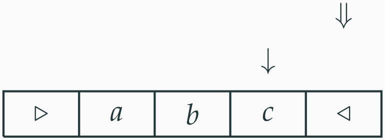


$$\texttt{squaring} : \Sigma^* \to (\Sigma \cup \underline{\Sigma})^*$$
$$aab \mapsto \underline{a}ab\,a\underline{a}b\,aa\underline{b}$$



output $= \underline{a}bca\underline{b}cab\underline{c}$

k-pebble transducers: executive summary

Finite set of states + a stack of two-way reading heads of height $\leq k$

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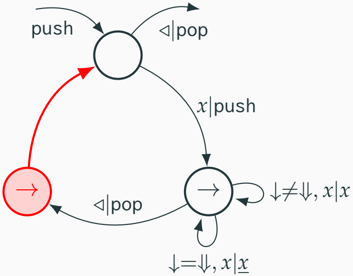

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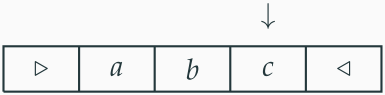

output $= \underline{a}bca\underline{b}cab\underline{c}$

k-pebble transducers: executive summary

Finite set of states + a stack of two-way reading heads of height $\leq k$

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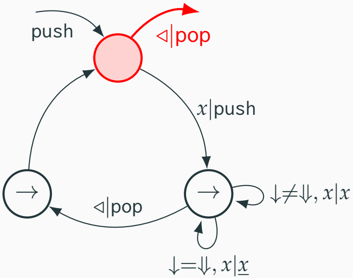
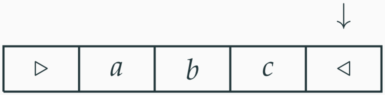


$$\texttt{squaring}: \quad \Sigma^* \to (\Sigma \cup \underline{\Sigma})^*$$
$$aab \mapsto \underline{a}ab\,a\underline{a}b\,aa\underline{b}$$



output $= \underline{a}bc\,a\underline{b}c\,ab\underline{c}$

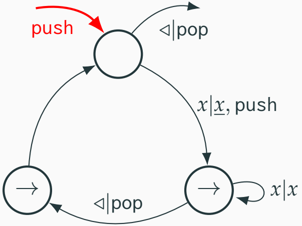
Comparison-free pebble transducers

Main question

What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring" $\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$



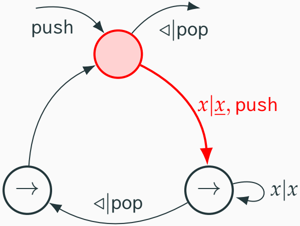
output =

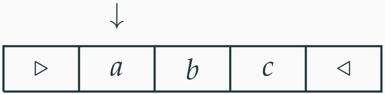
Main question

What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring" $\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





output =

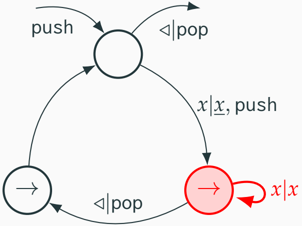
Main question

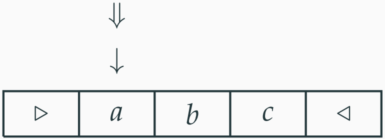
What happens if we disallow comparisons between reading heads?

Non-example: “squaring with underlining”

Example: “comparison-free squaring”

$\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





output $= \underline{a}$

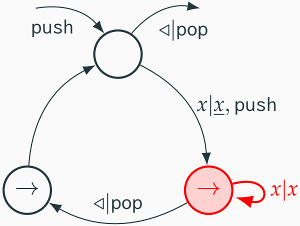
Main question

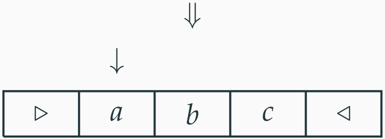
What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring"

$\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





output $= \underline{a}a$

Main question

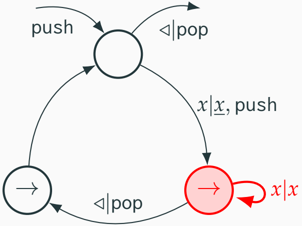
What happens if we disallow comparisons between reading heads?

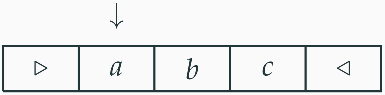
Non-example: "squaring with underlining"

Example: "comparison-free squaring"

`cfsquaring`$(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$

$\Downarrow$





output $= \underline{a}ab$

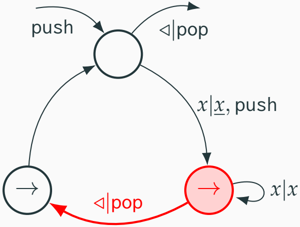
Main question

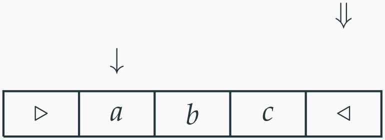
What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring"

$\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





output $= \underline{a}abc$

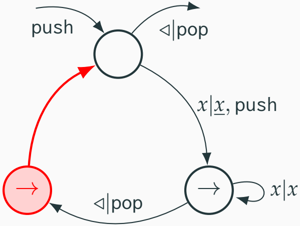
Main question

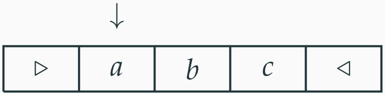
What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring"

$\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





output $= \underline{a}abc$

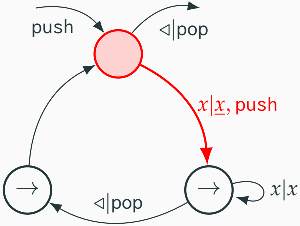
Main question

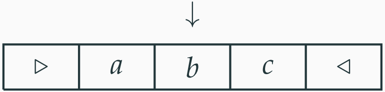
What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring"

$\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





output $= \underline{a}abc$

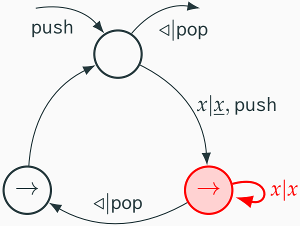
Main question

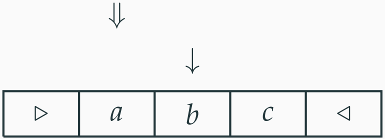
What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring"

$\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





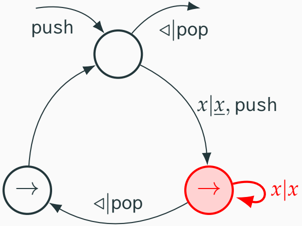
output $= \underline{a}abc\underline{b}$

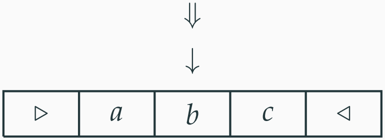
Main question

What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring" $\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





output $= \underline{a}abc\underline{b}a$

Main question

What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring"

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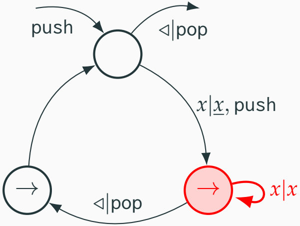

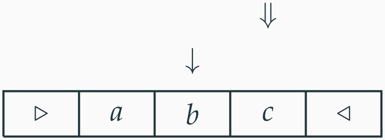

output $= \underline{a}abc\underline{b}ab$

Main question

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Example: "comparison-free squaring"

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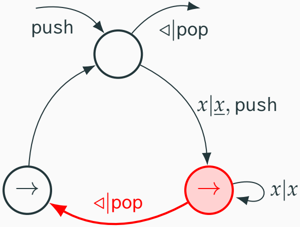

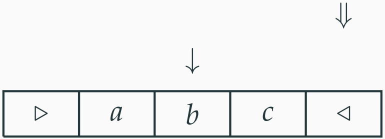

output $= \underline{a}abc\underline{b}abc$

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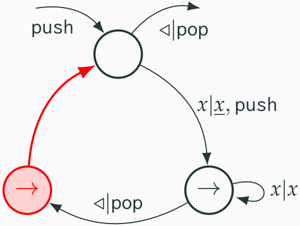

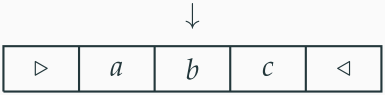

output $= \underline{a}abc\underline{b}abc$

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Non-example: “squaring with underlining”

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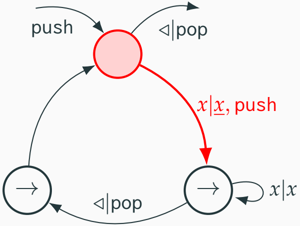

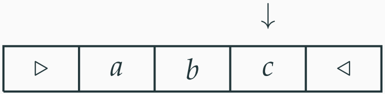

output $= \underline{a}abc\underline{b}abc$

Main question

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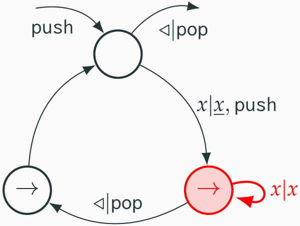

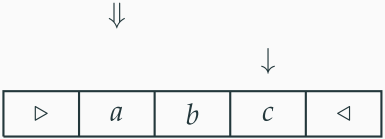

output $= \underline{a}abc\underline{b}abc\underline{c}$

Main question

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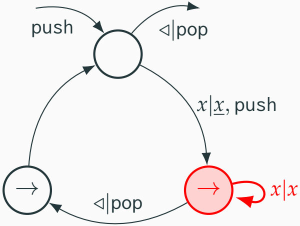
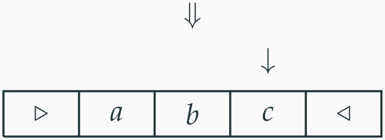




output $= \underline{a}abc\underline{b}abc\underline{c}a$

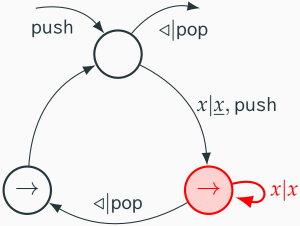
Main question

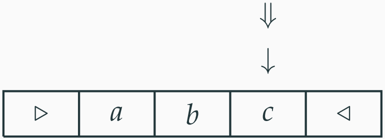
What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring"

$\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





$\text{output} = \underline{a}abc\underline{b}abc\underline{c}ab$

Comparison-free pebble transducers

Main question

What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring"

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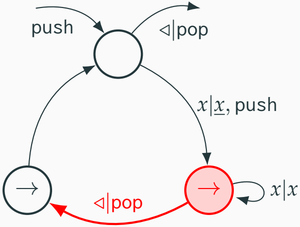

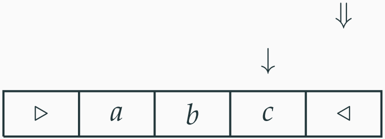

output $= \underline{a}abc\underline{b}abc\underline{c}abc$

Main question

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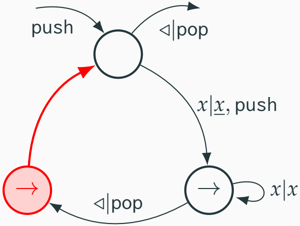

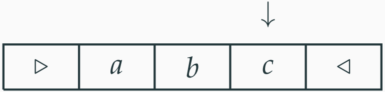

output $= \underline{a}abc\underline{b}abc\underline{c}abc$

Comparison-free pebble transducers

Main question

What happens if we disallow comparisons between reading heads?

Non-example: “squaring with underlining”

Example: “comparison-free squaring”

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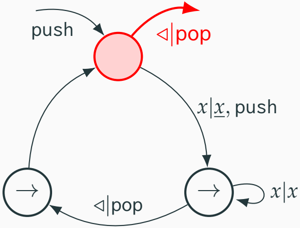
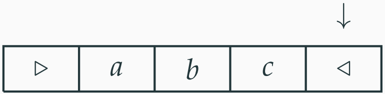




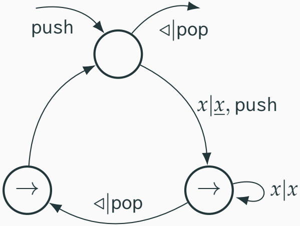
output $= \underline{a}abc\underline{b}abc\underline{c}abc$

Main question

What happens if we disallow comparisons between reading heads?

Non-example: "squaring with underlining"

Example: "comparison-free squaring" $\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$



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Contributions

- Alternative characterizations
- Separation results
- Along the way: closure by composition, pebble minimization

Some alternative characterizations

Alternative characterization (1/2): composition by substitutions

Definition (Composition by substitutions)

Let Γ, Σ and I be finite alphabets and $f : \Gamma^* \to I^*$, $g_i : \Gamma^* \to \Sigma^*$ and $w \in \Gamma^*$.
Define $\mathrm{CbS}(f, (g_i)_{i \in I})(w)$ so that, if $f(w) = i_1 \dots i_k$, then

$$\mathrm{CbS}(f, (g_i)_{i \in I})(w) = g_{i_1}(w) \dots g_{i_k}(w)$$

E.g. for `cfsquaring`, we take $f : \Sigma^* \to (\Sigma \cup \{X\})^*$, $g_X, g_a : \Sigma^* \to (\Sigma \cup \underline{\Sigma})^*$ (for $a \in \Sigma$) so that

$$f(abc) = aXbXcX \qquad g_a(w) = \underline{a} \quad \text{and} \quad g_X(w) = w$$

- Note: both cfpolyreg and polyregular functions are closed under CbS

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Alternative definition of cfpolyregular functions

Smallest class such that

- Every regular function is cfpolyreg
- If f is regular and g_i is cfpolyreg for every $i \in I$ then $\mathrm{CbS}(f, (g_i)_{i \in I})$ is cfpolyreg

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- More convenient to manipulate formally
- Tight link between the number of pebbles and the nesting of the CbS operator

Alternative characterization (2/2): composition and basic combinators

We have an alternative characterization based on linear the λ-calculus

- Not presented in the paper, mostly based on [Nguyễn,Noûs,P. 2020]
- Hints at the following non-trivial theorem

(reproven with automata-theoretic tools in the paper with no references to the λ-calculus)

Closure under composition

If $f : \Sigma^* \to \Gamma^*$ and $g : \Gamma^* \to \Delta^*$ are both cfpolyregular, so is $g \circ f$.

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Leads to a combinator-based definition.

Alternative definition of cfpolyregular functions

Least class containing the regular functions, `cfsquaring` and closed under composition.

- Analogous to the case of general polyregular functions

`cfsquaring` replaced by "squaring with underlining" in the above $\to$ all polyregular functions

- Regular functions can also themselves be decomposed

Not all polyregular functions are comparison-free

Separation results

Theorem

The function $f : a^n \in \{a\}^* \mapsto a\#aa\# \ldots \#a^n$
is polyregular but not comparison-free.

Corollary: "squaring with underlining" is not CF.

Theorem

$g : a^{n_1}\# \ldots \#a^{n_k} \in \{a,\#\}^* \mapsto a^{n_1 \times n_1}\# \ldots \#a^{n_k \times n_k}$
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$w \in \Gamma^* \mapsto w^{|w|}$ is comparison-free polyregular, but when $|\Gamma| \geq 2$, it is not HDT0L.

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For $h : \Gamma^* \to \Sigma^*$, $w_1, \ldots, w_n \in \Gamma^*$ with $\# \notin \Gamma$, $\mathbf{map}(h)(w_1\#\ldots\#w_n) = f(w_1)\#\ldots\#f(w_n)$.

$g = \mathbf{map}(w \mapsto w^{|w|})$ therefore comparison-free polyregular functions are *not* closed under **map**, unlike regular and polyreg functions

→ obstruction to characterizing cfpolyreg fn by list-processing functional programs (à la [Bojańczyk, Daviaud & Krishna 2018])

Separation proof idea for "map unary square"

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$g(a^{n_1}\# \dots \#a^{n_k}) = a^{n_1 \times n_1}\# \dots \#a^{n_k \times n_k}$ *is* not *comparison-free polyregular.*

Proof by contradiction: assume g is cfpolyreg.

First, $|g(w)| = O(|w|^2)$ *therefore* g is computed by some **2**-cf-pebble transducer.

Separation proof idea for "map unary square" via pebble minimization

Pebble minimization -- major result of our paper

If f is cfpolyreg and $|f(w)| = O(|w|^k)$ then some comparison-free k-pebble transducer computes f.

Very technical proof adapted from the analogous result for pebble transducers [Lhote 2020].

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Might be doable without pebble minimization, but convenient and of independent interest

Separation proof idea continued: unary inputs

Theorem

$f(a^n) = a\#aa\# \ldots \#a^n$ is not cfpolyreg.

Observation: $f(a^n)$ has the n maximal a-factors

$$a \quad aa \quad \ldots \quad a^n$$

Lemma

For any cfpolyreg $g : \{a\}^* \to \Sigma^*$, there are $O(1)$ possible lengths for maximal a-factors in $g(a^n)$.

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Smallest subclass of $(\Sigma^*)^{\mathbb{N}}$

- Containing the constant sequences $\alpha_n = w$
- Closed under concatenation $\alpha_n = \beta_n \cdot \gamma_n$
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- Regular word sequences are u.p.c. of pumping sequences $(u_0(v_1)^n \ldots (v_l)^n u_l)_{n\in\mathbb{N}}$ [Choffrut 2017]
  Proof idea: find an idempotent in a suitable transition monoid of your favorite machine model for reg fn
- Proof for general cfpolyreg sequences: induction on the CbS-based definition

Further topics

Going first-order

First-order (FO)-regular functions

Robust subclass of regular functions; several characterizations:

- Logic: replace MSO by first-order logic
- 2DFT with *aperiodic* monoid of behaviors
- Functional programming or regexp-like e.g. [Dartois, Gastin & Krishna 2021]

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Other characterizations?

Logical characterization of FO-cfpolyregular functions

Let $\mathfrak{M} : \{\text{words}\} \to \{\text{finite models}\}$ be as usual.
For $\mathfrak{U} = (U, R, \ldots)$, let $\mathfrak{U}^k = (U^k, R_1, \ldots, R_k, \ldots)$ where $R_i(x_1, \ldots, x_m) :\Leftrightarrow R(\pi_i(x_1), \ldots, \pi_i(x_m))$.

Conjecture

f is *first-order* comparison-free polyregular if and only there exist $k \in \mathbb{N}$ and a FO transduction φ s.t.

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- Equivalences with other candidates characterizing FO-cfpolyregular: apparently easier

E.g., FO-cfpolregular = closure under $\circ$ of FO-regular and `cfsquaring`

Further topics

A few relevant directions:

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- Non-commutative linear λ-calculus characterization for the FO case

Conclusion

Summary

A new(?) class of string-to-string functions: *comparison-free polyregular functions*.

Equivalent definitions

- By comparison-free pebble transducers
- Inductively (composition by substitution)
- **As the composition closure of regular functions +** `cfsquaring`$(abc) = \underline{a}abc\underline{b}abc\underline{c}abc$

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  - **pebble minimization theorem:** $k =$ **number of heads necessary to compute** f
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  - $a^n \mapsto a\#aa\#\ldots\#a^n$ not cfpolyreg but HDT0L
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Thanks for watching! We'll be happy to take questions