The logical complexity of MSO over countable linear orders

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Monadic Second-Order logic

Outline

Syntax of MSO

$$\varphi, \psi ::= R(t_1, \ldots, t_k) \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x\ \varphi \mid x \in X \mid \exists X\ \varphi$$

- Only *unary* predicates.
- The structures which we will discuss today:

the natural numbers

$(\omega, <)$

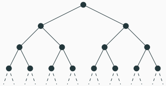
$\bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \cdots$

the rationals

$(\mathbb{Q}, <)$

the infinite (binary) tree

$(\{0,1\}^*, s_0, s_1, =)$



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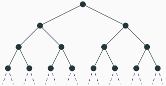

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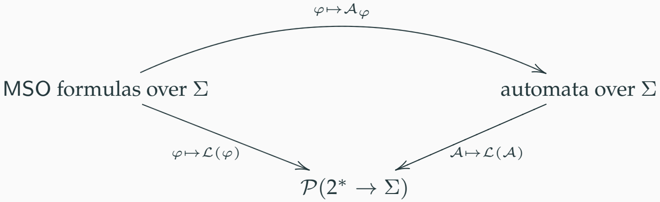
Typical MSO-definable properties

- ``The set X is unbounded.'' $(\omega, <)$
- ``There is no homomorphism $(\mathbb{Q}, <) \to (X, <)$ (i.e., X is *scattered*).'' $(\mathbb{Q}, <)$
- ``X intersects infinitely many times exactly one infinite branch.'' $(\{0,1\}^*, s_0, s_1, =)$

MSO/automata correspondance

Rabin's theorem (1971)

$\mathsf{MSO}(2^*, s_0, s_1, =)$ is decidable.



The high-level idea

- $\mathcal{L}(\varphi(X_1, \ldots X_n)) \subseteq [2^* \to 2^n]$ corresponds to the valuations $\{\rho \mid \mathsf{MSO}(\{0,1\}^*, s_0, s_1, =) \models_\rho \varphi\}$.
- Automata construction for each connective; $\exists$ and $\neg$ present the most difficulty.
- It is decidable to check whether $\exists t \in \mathcal{L}(\mathcal{A})$ or not.

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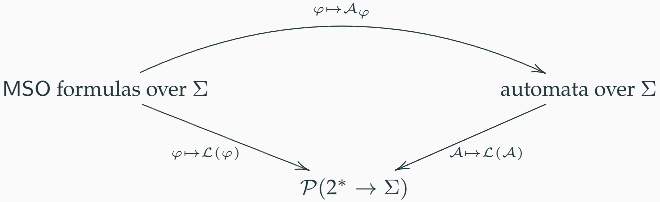

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- Decidability of $\mathsf{MSO}(\omega, <)$ and $\mathsf{MSO}(\mathbb{Q}, <)$ can be deduced from Rabin's theorem. (interpretations)
- Direct proof for $\mathsf{MSO}(\omega, <)$ using the same high-level approach (Büchi 1962).
- Assuming AC and CH, $\mathsf{MSO}(\mathbb{R}, <)$ is undecidable (Shelah 1975).

Automata

A non-deterministic word automaton $\mathcal{A} : \Sigma$ is a tuple (Q, q_0, δ, F) with

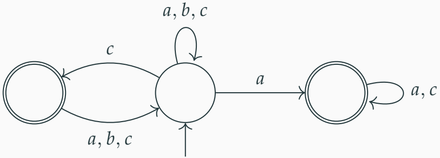
- Q is a finite set of states, $q_0 \in Q$
- a transition function $\delta : \Sigma \times Q \to \mathcal{P}(Q)$
- a set $F \subseteq Q$ of *accepting states*

A run over the input $w \in \Sigma^\omega$ is a sequence $\rho \in Q^\omega$ with $\rho_0 = q_0$ and $\forall n \in \omega\ \rho_{n+1} \in \delta(w_n, \rho_n)$

$$q_0 \xrightarrow{w_0} \rho_1 \in \delta(w_0, q_0) \xrightarrow{w_1} \rho_2 \in \delta(w_1, \rho_1) \xrightarrow{w_2} \dots$$

Büchi acceptance condition

$w \in \mathcal{L}(\mathcal{A}) \subseteq \Sigma^\omega$ iff there is a run over w hitting F infinitely often. non-recursive!



``There are infinitely many cs or finitely many bs.''

$$(\Sigma^* c)^\omega + \Sigma^*\{a, c\}^\omega$$

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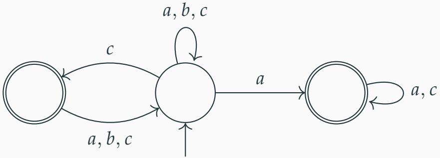

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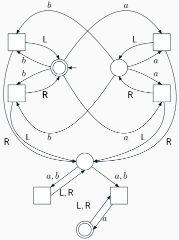

A tree automaton recognizing
``$\exists!$ branch with ∞ many bs''

Complement and projections

Major roadblocks toward proving the decidability theorems for $\mathsf{MSO}(\omega, <)$ and $\mathsf{MSO}(2^*, s_0, s_1, =)$

On ω-words

- For every Büchi automaton $\mathcal{A} : \Sigma$, there is $\mathcal{A}^c$ s.t. $\mathcal{L}(\mathcal{A}^c) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ (Büchi 1962)
- Büchi automata can be determinized into parity automata (McNaughton 1969)

Modern proofs typically involve weak König's lemma and infinite Ramsey for pairs

On labeled trees (Rabin 1971)

- For every non-deterministic parity tree automaton $\mathcal{A} : \Sigma$, there is $\mathcal{A}^c$ s.t. $\mathcal{L}(\mathcal{A}^c) = \Sigma^{2^*} \setminus \mathcal{L}(\mathcal{A})$
- *Alternating* parity tree automata $\equiv$ non-deterministic parity tree automata

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Motivating question

Those arguments are increasingly sophisticated from a combinatorial and logical perspective.
How can we quantify this?

Reverse Mathematics

- A framework to analyze axiomatic strength
- Vast program [Friedman, Simpson, Steele 70s]
- Many links with recursion theory

Methodology

- Consider a theorem T formulated in second-order arithmetic.
- Work in the weak theory RCA_0.
- Target some natural axiom A such that $\mathsf{RCA}_0 \nvdash A$.
- Show that $\mathsf{RCA}_0 \vdash A \Leftrightarrow T$.

Essentially independence proofs...

- Similar in spirit to statements like

``Tychonoff's theorem is equivalent to the axiom of choice.''

Induction and comprehension

RCA_0 is defined by restricting *induction* and *comprehension*

Comprehension axiom

For every formula $\phi(n)$ (with $X \notin FV(\phi)$)

$$\exists X \ \forall n \in \mathbb{N} \ [\phi(n) \Leftrightarrow n \in X]$$

- RCA_0: restricted to Δ^0_1 formulas — recursive comprehension

Induction axiom

To prove that $\forall n \in \mathbb{N} \ \ \phi(n)$ it suffices to show

- $\phi(0)$ holds
- for every $n \in \mathbb{N}$, $\phi(n)$ implies $\phi(n+1)$

- RCA_0: restricted to Σ^0_1 formulas — $\exists n \ \delta(n)$ with $\delta \in \Delta^0_1$
- Γ-induction equivalent to Γ-comprehension for finite sets

$$\forall n \in \mathbb{N} \ \ \exists X \ \ \forall k < n \ \ (k \in X \Leftrightarrow \phi(k))$$

The big five

Π^1_1 Comprehension	Π^1_1–CA_0	$\Longleftrightarrow$	Lusin's separation theorem
	$\Downarrow$		
Transfinite Recursion	ATR_0	$\Longleftrightarrow$	Determinacy of open games
	$\Downarrow$		
Σ^0_1 Comprehension	ACA_0	$\Longleftrightarrow$	König's Lemma
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Weak König's Lemma	WKL_0	$\Longleftrightarrow$	Brouwer's fixed point theorem
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Recursive Comprehension	RCA_0		

Outliers: infinite Ramsey for pairs, determinacy statements.

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Outliers: infinite Ramsey for pairs, determinacy statements.

$\rightsquigarrow$ Where do our decidability theorems sit in this hierarchy?

Between 2^* and ω: quick overview

Material covered in **How unprovable is Rabin's decidability theorem** [Kołodziejczyk, Michalewski, 2015]

Relationship to the big five

Complementation of non-deterministic tree automata and Rabin's theorem are

- provable in Π^1_3-comprehension
- unprovable in Δ^1_3-comprehension

$\rightsquigarrow$ well above Π^1_1-comprehension...

Main equivalence

Over ACA_0, the following are equivalent:

- Determinacy of $BC(\Sigma^0_2)$ games
- Positional determinacy of parity games
- Closure under complement of regular tree languages
- Decidability of $\mathsf{MSO}(2^*, s_0, s_1, =)$

Büchi's decidability theorem (over RCA_0)

Material covered in **The Logical Strength of Büchi's Decidability Theorem**

[Kołodziejczyk, Michalewski, P., Skrzypczak, 2016]

Weak König's lemma $\Downarrow$ Bounded weak König's lemma

Infinite Ramsey theorem $\Downarrow$

Σ^0_2-induction $\Longrightarrow$ Additive Ramsey

$\Uparrow$ $\Downarrow$

MSO(ω) $\Longleftarrow$ Compl. of NBA

MSO(ω) $\Downarrow$ Bounded weak König's lemma

Compl. of NBA $\Downarrow$ Determinization of NBA

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Compl. of NBA $\Longleftarrow$ ... MSO(ω) $\Longleftarrow$ Compl. of NBA

MSO(ω) $\Uparrow$ Σ^0_2-induction

MSO(ω) $\Downarrow$ Bounded weak König's lemma

Compl. of NBA $\Downarrow$ Determinization of NBA

Let's focus on additive Ramsey (main tool for complementation and algebraic approaches)

Additive Ramsey over ω

For any linear order $(P,<)$ write $[P]^2$ for $\{(i,j) \in P^2 \mid i < j\}$ and fix a finite monoid $(M,\cdot,e)$.

Call $f : [P]^2 \to M$ *additive* when $f(i,j) \cdot f(j,k) \ = f(i,k)$ for all $i < j < k$

Additive Ramsey

For any additive $f : [P]^2 \to M$, there is an unbounded monochromatic $X \subseteq P$ (s.t. $|f([X]^2)| = 1$).

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Π^0_2-induction from additive Ramsey

- Consider equivalently comprehension for sets bounded by n for $\exists^\infty k\;\; \delta(x,k)$

  (the set of infinite sets is a complete Π^0_2-set)
- Define the coloring $f : [\omega]^2 \to 2^n$ as $\;f(i,j)_x = \max\limits_{i\le l<j} \delta(x,l)$
- Apply additive Ramsey and consider the color X of the monochromatic set. Conclude as

$$x \in X \qquad \Longleftrightarrow \qquad \exists^\infty k\;\; \delta(x,k)$$

The big picture

$$
\begin{array}{ccccccccc}
\mathsf{WKL}_0 & \Leftarrow & \mathsf{ACA}_0 & \Leftarrow & \mathsf{ATR}_0 & \Leftarrow & \Pi^1_1-\mathsf{CA}_0 & & \\
& \swarrow\!\!\!\swarrow & & & & & & \nwarrow\!\!\!\nwarrow & \\
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$\mathsf{WKL}_0 \Leftarrow \mathsf{ACA}_0 \Leftarrow \mathsf{ATR}_0 \Leftarrow \Pi^1_1 - \mathsf{CA}_0$

$\mathsf{ACA}_0 \Rightarrow \Sigma^0_2 - \mathsf{IND} \Updownarrow \mathsf{MSO}(\omega, <)$

$\mathsf{MSO}(2^*, s_0, s_1, =) \Rightarrow \Delta^1_3 - \mathsf{CA}_0 \Rightarrow \Pi^1_1 - \mathsf{CA}_0$

$\Pi^1_3 - \mathsf{CA}_0 \Rightarrow \mathsf{MSO}(2^*, s_0, s_1, =)$

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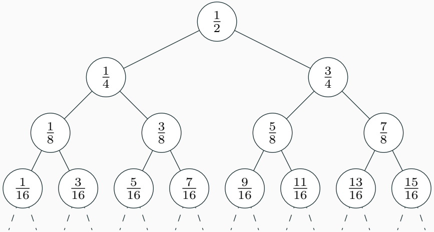
Motivates studying $\mathsf{MSO}(\mathbb{Q}, <)$ strictly intermediate?

Decidability of $\mathsf{MSO}(\mathbb{Q}, <)$ via algebras

Background on the decidability of $\mathsf{MSO}(\mathbb{Q},<)$

- Initially proven as a corollary of Rabin's theorem (other interesting examples also obtained like this)

$$\mathbb{Q} \quad \simeq \quad \left\{\frac{k}{2^n} \mid 1 \leq k \leq 2^n\right\} \quad \longmapsto$$



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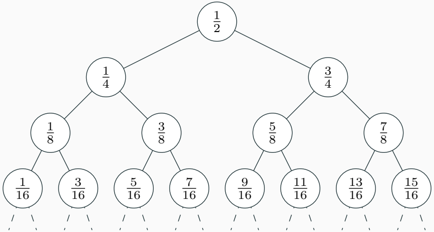

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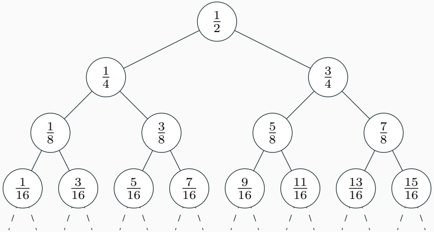

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  - By computing effectively (n, k)-types (n=quantifier depth and k=parameters)
  - In particular, coincides with the MSO theory of an Aronszajn line
  - Important subcase: scattered linear orders (no homomorphism $(\mathbb{Q}, <) \to (P, <)$)

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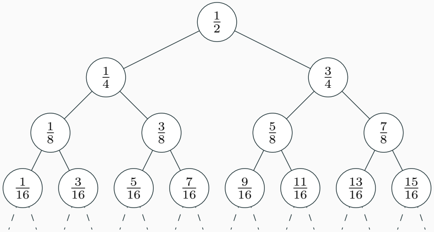

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- We will follow a modern presentation appearing in
  An algebraic approach to MSO-definability on countable linear orderings

  [O. Carton, T. Colcombet, G. Puppis, 2011]

Algebras for countable linear orders

Fix a set $\mathsf{LO}_{\aleph_0}$ containing all countable linear orders (up to iso) closed under *lexicograhic sums* $\sum_p Q_p$

$\circ$-monoid

A $\circ$-monoid is a pair $(M, (\mu_P)_{P \in \mathsf{LO}_{\aleph_0}})$ where

- M is a (finite) set
- $(\mu_P)_{P \in \mathsf{LO}_{\aleph_0}}$ is a family of maps $\mu_P : [P \to M] \to M$ that are *associative* (for $|P| \leq 2 \to$ monoid laws)

$$\begin{array}{ccc} \prod_{p \in P} [Q_p \to M] & \xrightarrow{\prod_p \mu_{Q_p}} & M^P \\ \sim \downarrow & & \downarrow \mu_P \\ [M \to \sum Q] & \xrightarrow[\mu_{\sum Q}]{} & M \end{array}$$

and stable under order-isomorphism

Algebras for countable linear orders

Fix a set $\mathsf{LO}_{\aleph_0}$ containing all countable linear orders (up to iso) closed under *lexicograhic sums* $\sum_p Q_p$

$\circ$-monoid

A $\circ$-monoid is a pair $(M, (\mu_P)_{P \in \mathsf{LO}_{\aleph_0}})$ where

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Typical examples: (n, r)-types of countable linear orders

Recognizing $\circ$-words

A countable word ($\circ$-word) over Σ is a map $P \to \Sigma$ with $P \in \mathsf{LO}_{\aleph_0}$

Recognition by $\circ$-monoids

Fix a finite alphabet Σ and a tuple (M, μ, φ, F) with

- (M, μ) a $\circ$-monoid
- $\varphi : \Sigma \to M$ and $F \subseteq M$

Say $w \in \Sigma^P$ is recognized by (M, μ, φ, F) iff $\mu_P(\varphi \circ w) \in F$

- Generalizes the algebraic approach to (in)finite word automata (recognition via (ω)-monoids)

Recognizing ∘-words

A countable word (∘-word) over Σ is a map $P \to \Sigma$ with $P \in \mathsf{LO}_{\aleph_0}$

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Challenges toward decidability

Find a finitary representation of $\circ$-monoids such that

- emptiness of a language restricted to domains $(\mathbb{Q}, <)$ may be checked algorithmically
- the powerset operation remains computable

Finitary presentation ○-algebra

○-algebra

A ○-algebra is a tuple $(M, \cdot, e, (-)^{\tau}, (-)^{\tau^{\mathsf{op}}}, (-)^{\kappa})$ where

- $(M, \cdot e)$ is a (finite) monoid
- the operations $(-)^{\tau}, (-)^{\tau^{\mathsf{op}}} : M \to M$ and $(-)^{\kappa} : \mathcal{P}(M) \setminus \emptyset \to M$ satisfy *associativity* equations

[omitted]

∘-algebra

A ∘-algebra is a tuple $(M, \cdot, e, (-)^{\tau}, (-)^{\tau^{\mathrm{op}}}, (-)^{\kappa})$ where

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Given an alphabet Σ, $a \in \Sigma$, $P \in \mathcal{P}(\Sigma) \setminus \emptyset$ write

- a^{ω} and $a^{\omega^{\mathrm{op}}}$ for the constant maps $- \mapsto a$ with domain ω and ω^{op}

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  We call these words K-shuffles

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A ∘-monoid maps to a ∘-algebra by setting $a^{\tau} = \mu_{\omega}\left(a^{\omega}\right), a^{\tau^{\mathrm{op}}} = \mu_{\omega^{\mathrm{op}}}\left(a^{\omega^{\mathrm{op}}}\right)$ and $P^{\kappa} = \mu_{\mathbb{Q}}\left(P^{\eta}\right)$

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Theorem (representability)

Every finite ∘-algebra has a unique lift to a ∘-monoid.

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Every finite $\circ$-algebra M has a unique lift to a $\circ$-monoid.

A convex subset $Q \subseteq_{\text{conv}} P$ is a set $Q \subseteq P$ such that $x, y \in Q \wedge x < z < y \implies z \in Q$

Say that a countable word $w : P \to M$ has value m if there is an associative

$$\mu \; : \; \prod_{Q \subseteq_{\text{conv}} P} \left[M^Q \to M \right]$$

compatible with M such that $\mu_P(w) = m$

Outline of the argument

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3. ${}^{P}/_{\sim}$ is necessarily a subsingleton
   - If two successive elements in ${}^{P}/_{\sim}$, contradiction because of binary multiplication
   - Otherwise, ${}^{P}/_{\sim}$ is dense and there is a shuffle in ${}^{w}/_{\sim}$, contradiction because of $(-)^{\kappa}$

The additional fine combinatorial ingredient: shuffle principle/additive Ramsey over $\mathbb{Q}$

The shuffle principle

For any $n \in \mathbb{N}$ and $c : \mathbb{Q} \to n$, there is $I \subseteq_{\text{conv}} \mathbb{Q}$ such that $c \restriction I$ is a shuffle.

Compare and contrast with the key combinatorial principle in Shelah's argument

Shelah's additive Ramseyan theorem

For every additive map $f : [\mathbb{Q}]^2 \to M$, there exists

- $I \subseteq_{\text{conv}} \mathbb{Q}$
- finitely many dense sets D_i with $I = \bigcup_i D_i$

such that f is constant over each $[D_i]^2$

Decidability

Powerset $\circ$-monoid

Define the operation $(M, \mu) \mapsto (\mathcal{P}(M), \mu^{\mathcal{P}})$ as

$$\mu_P^{\mathcal{P}}(w) = \{\mu(u) \mid u \in M^P, \forall x \in P \;\; u(x) \in w(x)\}$$

This $\circ$-monoid is important as allows to produce

- A tuple $(\mathcal{P}(M), \mu^{\mathcal{P}}, \varphi^{\exists}, F^{\exists})$ recognizing a projection of $\mathcal{L}(M, \mu, \varphi, F)$
- Go from the $(n, k+1)$-types to $(n+1, k)$-types

Powerset ∘-monoid

Define the operation $(M, \mu) \mapsto (\mathcal{P}(M), \mu^{\mathcal{P}})$ as

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Lemma

The underlying map of ∘-algebra is computable

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Lemma

The underlying map of $\circ$-algebra is computable

Corollary

$\mathsf{MSO}(\mathbb{Q}, <)$ is decidable

Reverse Mathematics of $\mathsf{MSO}(\mathbb{Q}, <)$

The fine combinatorial principles?

Do the more obvious combinatorial principles contribute to the logical complexity once again?
Not really

Theorem

Over RCA_0, the following are equivalent:

- the shuffle principle
- Shelah's additive Ramseyan theorem over $\mathbb{Q}$
- induction for Σ^0_2 formulas

(Recall that $\mathsf{RCA}_0 \wedge \mathsf{MSO}(\mathbb{Q}, <) \implies \Pi^1_1\mathsf{CA}_0$)

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(Recall that $\mathsf{RCA}_0 \wedge \mathsf{MSO}(\mathbb{Q}, <) \implies \Pi^1_1\mathsf{CA}_0$)

The implications $\implies \Sigma^0_1 - \mathsf{IND}$ are proven similarly as before using the map

$$\begin{array}{ccc} \{\frac{2k+1}{2^n} \mid 0 \leq k \leq 2^{n-1}\} & \longrightarrow & \mathbb{N} \\ \frac{2k+1}{2^n} & \longmapsto & n \end{array}$$

density $\Longleftarrow$ infinity

An upper bound and a conjectural upper bound

Adapting the approach above, with the following caveats:

- Some lemmas cannot be stated in the language of second-order arithmetic as-is

  (adapted statements: talk about infinitary syntax trees and algebras only)
- Swept the effectivization of $(\mathcal{P}(M), \mu^{\mathcal{P}})$ under the rug (needs to be reformulated anyways)
- We would at several points use conservativity of choice for certain classes of formualas

Theorem

Π^1_2-comprehension proves decidability of $\mathsf{MSO}(\mathbb{Q}, <)$

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Theorem

Π^1_2-comprehension proves decidability of $\mathsf{MSO}(\mathbb{Q}, <)$

- This shows that this is strictly easier than Rabin's theorem, strictly harder than Büchi's
- We have reasons to suspect this is not optimal

Operating conjecture

The axiom of finite Π^1_1-recursion ($\phi \in \Pi^1_1$, $X \notin FV(\phi)$)

$$\forall n\, \exists X. X_0 = \emptyset \wedge \forall k < n\, \forall z\, (z \in X_{k+1} \Leftrightarrow \phi(z, X_k))$$

- Always true in *standard* models of $\Pi^1_1 - \mathsf{CA}_0$.
- This is equivalent to determinacy of weak parity games $BC(\Sigma^0_1)$ GS games

Conjecture

Finite Π^1_1-recursion proves the soundness of the standard decision algorithm for $\mathsf{MSO}(\mathbb{Q})$

- So far, we know how to prove the analogue of the representation lemma
- We miss the soundness of the definition of the powerset algebra
- Enough to derive a descriptive set theoretic result

Evaluating words with finite Π^1_1-recursion (scattered vs dense)

Now let us sketch the argument for a representability theorem. Fix a $\circ$-algebra M.
Consider the following procedure to compute the value of a word $w : P \to M$

Iterate the following two steps

1. When P is dense in itself, factorize *pseudo-shuffles* maximally
2. Otherwise, decompose P as a sum of *scattered orders* and evaluate each scattered part

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Hausdorff's theorem $\Pi^1_1 \to$ **(Clote 1989)**

Every linear order is isomorphic to a Π^1_1-definable decomposition $\sum_{d \in D} P_d$ where

- D is dense in itself (if countable, either $0, 1$ or $\mathbb{Q}$ up to endpoints)
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- Recursion over a decomposition of P along a well-founded ordered trees with arities $\subseteq \mathbb{Z}$
- Relies on the arithmetical definition of monochromatic sets for additive Ramsey

Evaluating words with finite Π_1^1-recursion (dense steps)

Consider the following procedure to compute the value of a word $w : P \to M$

Iterate the following two steps

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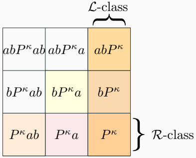
2. Otherwise, decompose P as a sum of *scattered orders* and evaluate each scattered part

Pseudo-shuffles

$w : \mathbb{Q} \to M$ is a pseudo-shuffle of value $e \in M$ if:

- for each convex subword which is a P-shuffle, we have $P^\kappa = e$
- for every letter m occuring in w, $eme = e$
- for each homomorphism $\iota : \mathbb{Q} \to \mathbb{Q}$ such that $w \circ \iota$ is a P-shuffle, $(P \cup \{e\})^\kappa = e$

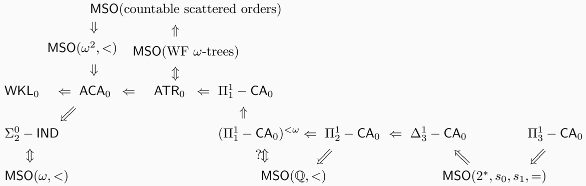
- More general than shuffles
- Note the dependency on the structure of M
- Required to bound the number of iterations by $|M|$
- Algebraic reasoning on $\circ$-algebras needed



(compatibility with the monoid structure)

Conclusion

The current picture



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- ...but we do not have a clean equivalence
- Improved characterization of ∘-word languages in terms of topological complexity?

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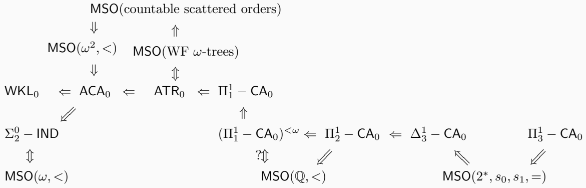

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Conjecture on MSO-definable languages

Define the C-hierachy by iterating Suslin A-operation and complementation $(\Sigma^1_1 \subseteq \mathrm{C} \subsetneq \Delta^1_2)$

Every $\mathsf{MSO}(\mathbb{Q}, <)$-definable language sits in a finite level of the C-hierarchy

(beforehand, Δ^1_2 bound via a collapse result in (Carton, Colcombet, Puppis 2011))

Further questions

- Settle the conjectures!
- Characterize algebras recognizing Borel languages
- Are well-founded trees strictly harder than scattered words/countable ordinals?
- Logical strength related to weak parity games
  - $\rightsquigarrow$ Is there a natural alternating automata model for $\mathbb{Q}$-labellings?
- Adapt the techniques for uncountable structures

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- Are well-founded trees strictly harder than scattered words/countable ordinals?
- Logical strength related to weak parity games
  - $\rightsquigarrow$ Is there a natural alternating automata model for $\mathbb{Q}$-labellings?
- Adapt the techniques for uncountable structures

Thanks for listening! Further questions?

MSO$(\mathbb{Q}, <)$ and C-sets

Fix a Polish space X. Note in particular that the set of words $\Sigma^{\mathbb{Q}}$ always forms a Polish space

(via $\mathbb{N} \simeq \mathbb{Q}$)

C-sets

Suslin A-operation takes a map $\beta : \mathbb{N}^* \to \mathcal{P}(X)$ and outputs the set

$$A(\beta) = \bigcup_{b \in \mathbb{N}^{\mathbb{N}}} \bigcap_{k \in \mathbb{N}} \beta(p \restriction k)$$

Extend the A operation to pointclasses $\Gamma \subseteq \mathcal{P}(X)$ by setting $A(\Gamma) = \{A(\beta) \mid \beta : \mathbb{N}^* \to \Gamma\}$

C-sets are obtained by iterating the A-operation from the closed sets and closing under complement

We have that $A(\Pi^0_1) = \Sigma^1_1$ and that C-sets are all Δ^1_2

Conjecture on MSO-definable languages

Every MSO$(\mathbb{Q}, <)$-definable language sits in a finite level of the C-hierarchy

For every finite level of the hierarchy of C-sets, there is a complete MSO$(\mathbb{Q}, <)$-definable language

- The first point is the more difficult result
- The second requires (already known) tricks to encode lexicographic products $\mathbb{Q} \times_{\text{lex}} \mathbb{Q}$