Represented spaces of represented spaces

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Abstract. We investigate spaces of spaces in the category of represented spaces and type-2 computable maps. Concretely, such spaces of spaces are given by bundles whose bases are names for spaces. We give natural examples of such bundles for Polish and compact Polish spaces and show they are essentially equivalent. We then propose definitions of genericity and computable categoricity, according to which the Cantor space is computably categorical and generic as a compact Polish space. We also show that the degree of categoricity of $\mathcal{S}^1$ is the Weihrauch degree lim.

Keywords: Computable topology · categoricity.

1 Introduction

We want to discuss notions of represented spaces of spaces so that we may formalize some natural questions such as

- what is a generic X-space? (for X ranging over various adjectives such as Polish, QuasiPolish, compact Polish...)
- is a space computably categorical? If not, what is the (recursion-theoretic) complexity of computing an isomorphism?

But first, we need to explain what a space of spaces is. Describing a space of spaces involves coding names for spaces as a space and giving an interpretation for them consisting of a bundle over the space of names. We are doing this in the category ReprSp of represented spaces and type-2 computable maps.

Concretely, such bundles are simply computable maps $\mathrm{El}_A : A_\bullet \to A$ between represented spaces $A_\bullet$ and A. The points of A are codes for the spaces we want to represent and $A_\bullet$ is a sum of those spaces. A point $x \in A$ represents the fiber $\mathrm{El}_A^{-1}(x)$, so the map El_A should be thought of as a projection

$$\begin{array}{rcl} \sum_{x \in A} \mathrm{El}_A^{-1}(x) \;\cong\; A_\bullet & \longrightarrow & A \\ (\text{a code } x \text{ for a space } S_x \cong \mathrm{El}_A^{-1}(x), \text{ a point of } S_x) & \longmapsto & x \end{array}$$

This in particular induces a map $\mathrm{El}_A^{-1} : A \to \mathsf{ReprSp}$, but also carries additional information about how the individual spaces $\mathrm{El}_A^{-1}(x)$ cohere together in $A_\bullet$, which allows us to talk about, say, maps that take as input a name x for a non-empty space and return a point of $\mathrm{El}_A^{-1}(x)$, *computably in* x. This is a standard approach to internalizing the notion of families in categories with pullbacks [11,19].

Example 1. One bundle representing finite-dimensional Euclidean spaces is the first projection of vectors of real numbers onto their dimensions $\mathbb{R}^* \to \mathbb{N}$, where $\mathbb{R}^*$ is represented by $\delta_{\mathbb{R}^*}(\langle n, x\rangle) = \langle \delta_{\mathbb{N}}(n), \delta_{\mathbb{R}^n}(x)\rangle$, with $\delta_{\mathbb{N}}, \delta_{\mathbb{R}^n}$ being standard representations for $\mathbb{N}$ and $\mathbb{R}^n$.

Example 2. A hyperspace $\mathcal{H}$ over some space X is a represented space whose points are subsets of X. An example is the hyperspace $\mathcal{A}(X)$ of closed subsets of X.

Any hyperspace over X induces a bundle $\mathcal{H}_\bullet \xrightarrow{\pi_1} \mathcal{H}$ where $\mathcal{H}_\bullet$ is the subspace of $\mathcal{H} \times X$ consisting of those pairs (A, x) such that $x \in A$.

We first investigate bundles that represent Polish spaces and compact Polish spaces. There are several natural options, which we show to be equivalent in a suitable sense (Section 2). We then define what it means for a space represented in such bundles to be computably categorical and, more generally, what the Weihrauch degree of categoricity of a space is. We then show that $\{0,1\}^{\mathbb{N}}$ is computably categorical in the compact Polish spaces and that the degree of categoricity of the circle $\mathcal{S}^1$ is lim (Section 3). Finally we define what it means for a space to be generic in a bundle and show that $\{0,1\}^{\mathbb{N}}$ is Π^0_2-generic in the compact Polish spaces (Section 4).

Related Work An early occurrence of the idea of a represented space of (certain) represented spaces is found in [17], where the space of Polish spaces is introduced. A representation of QuasiPolish spaces as an internal category of represented spaces was investigated by de Brecht [3], who shows that much of the usual structure over the category of QuasiPolish spaces does computably internalize.

The space $\mathcal{C}([0,1])$ is not computably categorical as a Banach space [14, Corollary 4.3] (up to isometry). An early contribution to the question of computable categoricity is [13], where it is e.g. shown that $\{0,1\}^{\mathbb{N}}$ is computably categorical in the sense of isometry.

Rather than looking at represented spaces of spaces, another approach is to investigate numberings of computable spaces of that kind. This approach is e.g. taken in [5].

Degrees of categoricity of various Lebesgue spaces have been extensively investigated [12,7].

2 Spaces of spaces as bundles

2.1 Equivalence of bundles

As previously mentioned, a bundle is simply any map $\mathrm{El}_A : A_\bullet \to A$ that we want to interpret as a space of spaces.

Given two bundles El_A and El_B, we can define what it means for El_A to be essentially equivalent to El_B by using the notion of reindexing and pullbacks.

A pullback square is a commuting square as found in the diagram below, satisfying the following universal property: if there are morphisms α, β as depicted such that $f \circ \beta = g \circ \alpha$, then there is a unique γ such that $\alpha = k \circ \gamma$ and $\beta = h \circ \gamma$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!\gamma"'] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& A \times_C B \arrow[r, "k"'] \arrow[d, "h"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

In this diagram $A \times_C B$ (together with the projections h and k) is a pullback of f and g. We use the notation around the top-left corner of the commuting square to indicate we intend this to be a pullback in a diagram. Pullbacks are only determined up to unique isomorphisms. Concretely, pullbacks in all categories we are interested in here can be built by taking the cartesian products of the domains of the maps f and g and restricting to the sets of pairs (a, b) with $f(a) = g(b)$.

We say that El_A is a pullback of El_B by $f : A \to B$ when there is a map $f_\bullet : A_\bullet \to B_\bullet$ such that the square below is a pullback.

$$\begin{tikzcd}
A_\bullet \arrow[r, dotted, "f_\bullet"] \arrow[d, "\mathrm{El}_A"'] \arrow[dr, phantom, "\lrcorner\ \exists", very near start] & B_\bullet \arrow[d, "\mathrm{El}_B"] \\
A \arrow[r, dotted, "f"'] & B
\end{tikzcd}$$

Essentially, f maps A-codes of spaces to B-codes of spaces and $f_\bullet$ gives us maps $\mathrm{El}_A^{-1}(a) \to \mathrm{El}_B^{-1}(f(a))$, uniformly in a. The square being a pullback moreover ensures that those maps are actually homeomorphisms whose inverses are also computable from codes $a \in A$. To define equivalence of two such bundles, we do not care if A and B are not isomorphic as long as El_A and El_B describe the same spaces and we can effectively translate a name for a space in A to a name for the same space in B and vice-versa. Hence we take the following definition for equivalence of bundles.

Definition 1. *We call two bundles (weakly) equivalent when they are pullbacks of one another.*

2.2 Intensional vs extensional equivalence of bundles

Sometimes we do not want to work extensionally and work with multi-valued maps. However, the notion of bundle in the category of multi-valued computable functions does not do what we want, so we are doing something else to deal with that difficulty: we replace a represented space by the subspace of Baire space consisting of the names of its points. These spaces can actually be characterized up to homeomorphism as the regular projective spaces of the category of represented spaces.

Definition 2. *Let us call a computable map between represented spaces a* quotient map *if it is has a computable multi-valued right-inverse.*

A projective reindexing of a bundle $\mathrm{El}_B : B_\bullet \to B$ *is a bundle* $\mathrm{El}_A : A_\bullet \to A$ *which is a pullback of* El_B *along a quotient map* $e : A \twoheadrightarrow B$ *such that* A *is the domain of a representation of* B*.*

By our discussion above, any bundle has at least one projective reindexing determined by the representation map of A and a choice of pullbacks. We avoid defining this as "the" projective redindexing as we want our constructions to be closed under bundle equivalence.

Lemma 1. *Any two projective reindexings of a bundle are equivalent.*

Definition 3. *We say that two bundles are* intensionally equivalent *if any of their respective projective reindexings are equivalent.*

2.3 Bundles of all Polish spaces and compact Polish spaces

We now consider several bundles meant to represent all (non-empty) Polish spaces.

We can code Polish spaces in the subspace PM of $\mathbb{R}^{\mathbb{N}^2}$ consisting of the pseudometrics over $\mathbb{N}$. Then the bundle is defined as $\mathrm{PM}_\bullet \xrightarrow{\pi_1} \mathrm{PM}$ where $\mathrm{PM}_\bullet$ is the obvious quotient of the subspace of $\mathrm{PM} \times \mathbb{N}^{\mathbb{N}}$ consisting of those pairs (d, s) such that s is a fast-converging sequence with respect to d.

The other alternatives are induced by hyperspaces as per Example 2. We write $\mathcal{A}(X)$, $\mathcal{V}(X)$ and $\Pi^0_2(X)$ for the hyperspaces of closed, overt and Π^0_2 subspaces of X (see [15], [9] for definitions). Given a hyperspace $\mathcal{H}$ over some set, we write $\mathcal{H}_+$ for its restriction to non-empty subspaces. Given $\mathcal{H}$ and $\mathcal{H}'$ over the same set, we write $\mathcal{H} \wedge \mathcal{H}'$ for the join hyperspace determined by the subspace of $\mathcal{H} \times \mathcal{H}'$ of pairs (A, A).

Theorem 1. *The bundle* $\mathrm{PM}_\bullet \xrightarrow{\pi_1} \mathrm{PM}$ *is intensionally equivalent to each of the bundles generated by* $(\Pi^0_2 \wedge \mathcal{V})([0,1]^\omega)_+$*,* $(\mathcal{A} \wedge \mathcal{V})(\mathbb{R}^\omega)_+$ *and* $\mathcal{V}(\mathbb{R}^\omega)_+$*.*

Lemma 2. *For a computable metric space* $\mathbf{X}$ *the computable map* $f \mapsto f^{-1}(\{0\}) : \mathcal{C}(\mathbf{X}, [0,1]) \to \mathcal{A}(\mathbf{X})$ *has a computable multivalued right-inverse.*

Corollary 1. *For a computable metric space* $\mathbf{X}$ *the computable map* $f \mapsto f^{-1}((0,1]^\omega) : \mathcal{C}(\mathbf{X}, [0,1]^\omega) \to \Pi^0_2(\mathbf{X})$ *has a computable multivalued right-inverse.*

Proof (Theorem 1). We show that some of the bundles in the theorem statement, or projective reindexings thereof, are pullbacks of one another so that we end up with a cycle of pullbacks witnessing that those bundles are pairwise intensionally equivalent.

For each direction, we only describe the maps between the bases of the bundles, leaving the reader to complete the top part of the pullbacks in the obvious way.

- PM is a pullback of $(\Pi^0_2 \wedge \mathcal{V})([0,1]^\omega)_+$:

  Without loss of generality, we can assume that the pseudometrics are bounded by 1. We then map a pseudometric $d : \mathbb{N} \times \mathbb{N} \to [0,1]$ to

  $$\{x \in [0,1]^\omega \mid \forall k \; \exists m \; \forall i \leq k \; |x_i - d(i,m)| < 2^{-k}\}$$

  This immediately yields a Π^0_2-name for the set, and we obtain the overt information by observing that a basic open set $U_0 \times \ldots \times U_\ell \times [0,1]^\omega$ intersects the set if and only if $\exists m \in \mathbb{N} \; \forall i \leq \ell \; d(i,m) \in U_i$.
- Projective reindexings of $(\Pi^0_2 \wedge \mathcal{V})([0,1]^\omega)_+$ are pullbacks of $(\mathcal{A} \wedge \mathcal{V}_+)(\mathbb{R}^\omega)_+$:
  Given some $A \in \Pi^0_2([0,1]^\omega)$, we can compute some $f : [0,1]^\omega \to [0,1]^\omega$ such that $A = f^{-1}((0,1]^\omega)$ by Corollary 1. This means that $\frac{1}{f} : A \to \mathbb{R}^\omega$ is well-defined, where the multiplicative inverse is taken component-wise, and we can compute $\frac{1}{f}$ from A. We then map A to $D_f := \left\{\left(x, \frac{1}{f}(x)\right) \mid x \in A\right\} \subseteq \mathbb{R}^\omega \times \mathbb{R}^\omega$. Since $(x,y) \notin D_f \Leftrightarrow \left(y \geq 1^\omega \wedge f(x) = \frac{1}{y}\right)$, we obtain $D_f \in \mathcal{A}(\mathbb{R}^\omega)$. Moreover, if we have $A \in \mathcal{V}([0,1]^\omega)$, we can obtain $D_F \in \mathcal{V}(\mathbb{R}^\omega)$, since overt sets are closed under continuous images.
- $(\mathcal{A} \wedge \mathcal{V})(\mathbb{R}^\omega)_+$ is a pullback of $\mathcal{V}(\mathbb{R}^\omega)_+$:

  We can just forget the closed information.
- Projective reindexings of $\mathcal{V}_+(\mathbb{R}^\omega)$ are pullbacks of PM:

  We can use overt choice to find a dense sequence $(x_n)_n$ in the given overt set, and then obtain the pseudometric $d(n,m) = d_{\mathbb{R}^\omega}(x_n, x_m)$.

Given a pseudometric d over $\mathbb{N}$, a witness of total boundedness is a function $t : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $x \in \mathbb{N}$, there exists $y < t(k)$ such that $d(x,y) < 2^{-k}$. Call $\mathrm{TBPM} \subseteq \mathbb{N}^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{N}}$ the subset consisting of pseudometrics and witnesses of total boundedness and $\mathrm{TBPM}_\bullet \xrightarrow{\pi_1} \mathrm{TBPM}$ the bundle obtained by pulling back $\mathrm{PM}_\bullet \xrightarrow{\pi_1} \mathrm{PM}$ along the first projection $\mathrm{TBPM} \to \mathrm{PM}$.

Theorem 2. *The bundle* $\mathrm{TBPM}_\bullet \xrightarrow{\pi_1} \mathrm{TBPM}$ *is intensionally equivalent to the bundle induced by* $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)_+$.

Proof. To show that $\mathrm{TBPM}_\bullet \xrightarrow{\pi_1} \mathrm{TBPM}$ is a pullback of the bundle induced by $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)_+$, we can use the same approach as in the proof of Theorem 1 by mapping the pair (d,t) of a pseudometric and a witness of total boundedness to the set

$$\{x \in [0,1]^\omega \mid \forall k \; \exists m \leq t(k) \; \forall i \leq k \; |x_i - d(i,m)| < 2^{-k}\}$$

This set is obviously effectively closed in $[0,1]^\omega$, so it is in particular effectively compact since $[0,1]^\omega$ is effectively compact. Note that, by the definition of witness of total boundedness, this set is the same as the set defined in the proof of Theorem 1; we can therefore get the overt information in the same way.

To show that any projective reindexing of $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)_+$ is a pullback of $\mathrm{TBPM}_\bullet \xrightarrow{\pi_1} \mathrm{TBPM}$, we can use overt choice as before to find a dense sequence $(x_n)_n$ with which to build a pseudometric over $\mathbb{N}$. The corresponding witness of total boundedness can then be reconstructed using the compact information.

3 Computable categoricity

We can also quantify the hardness of computing isomorphisms.

Definition 4. *Given a represented space S, the degree of categoricity* $\mathrm{CCat}(S, \mathrm{El}_A)$ *of S in a bundle $\mathrm{El}_A : A_\bullet \to A$ is given by the following Weihrauch problem:*

- ***Input:*** $(a, b) \in A^2$ *such that S, $\mathrm{El}_A^{-1}(a)$ and $\mathrm{El}_A^{-1}(b)$ are homeomorphic*
- ***Output:*** *a homeomorphism witnessing* $\mathrm{El}_A^{-1}(a) \cong \mathrm{El}^{-1}(b)$

We say that S is computably categorical when $\mathrm{CCat}(S, \mathrm{El}_A) \leq_W id$. We omit El_A when it is clear from context.

For any S and bundle El_A, note that $\mathrm{CCat}(S, \mathrm{El}_A)$ is at most Σ^1_1.

Proposition 1. *For any space S and two intensionally equivalent bundles El_A and El_B, $\mathrm{CCat}(S, \mathrm{El}_A) \equiv_W \mathrm{CCat}(S, \mathrm{El}_B)$.*

Proof. Trivial.

Theorem 3. $\{0,1\}^\mathbb{N}$ *is computably categorical as a compact Polish space.*

Proof (Sketch). Given a set $X \in (\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$ that is homeomorphic to Cantor space, we can search for a cover of X by two basic open sets U_1, U_2, such that each U_i intersects X and the closures of U_1 and U_2 are disjoint. This search must succeed since X is homeomorphic to Cantor space. Now, the closures of U_1 and U_2 must again be homeomorphic to Cantor space, so that we can iterate this process, which allows us to construct an explicit isomorphism with the standard Cantor space $\{0,1\}^\mathbb{N}$.

The degree of categoricity of the circle

As a somewhat more complicated example, we will investigate the degree of categoricity of the circle $\mathcal{S}^1$. We establish a slightly more general result, using the notation $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)|_{\mathrm{TC}}$ to denote the restriction of $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$ to sets which are classically homeomorphic to $\mathcal{S}^1$.

Theorem 4. *The following operations are Weihrauch equivalent:*

1. lim.
2. $\text{Image}^{-1} : (\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)|_{\text{TC}} \rightrightarrows \mathcal{C}(\mathcal{S}_1, [0,1]^\omega)$ *mapping* S *to some* $f : \mathcal{S}_1 \to [0,1]^\omega$ *with* $f[\mathcal{S}_1] = S$.
3. $\text{Trace} : (\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)|_{\text{TC}} \rightrightarrows \mathcal{C}(\mathcal{S}_1, [0,1]^\omega)$ *mapping* S *to some injective* $f : \mathcal{S}_1 \to [0,1]^\omega$ *with* $f[\mathcal{S}_1] = S$.

Corollary 2. $\text{CCat}(\mathcal{S}^1) \equiv_W \lim$.

Proof. A computable bijection between computably compact computably Hausdorff spaces is already a computable isomorphism.

The proof of the theorem follows later, after we have gathered a few crucial lemmas.

Definition 5. *For continuous* $f : S^1 \to [0,1]^\omega$ *and two disjoint open balls* $B_1, B_2 \in \mathcal{O}([0,1]^\omega)$, *we say that* f *oscillates between* B_1 *and* B_2 *at least* n *times if and only if there are* $t_1^1 < t_1^2 < t_2^1 \ldots < t_n^1 < t_n^2 \in [0,1]$ *such that* $f(t_i^j) \in B_j$ *(we view* S^1 *as a quotient of* $[0,1]$ *here). We say that* f *oscillates at most* n *times if it does not oscillate at least* $n+1$ *times.*

Lemma 3. *Given* $f \in \mathcal{C}(S^1, [0,1]^\omega)$, $x_1, x_2 \in [0,1]^\omega$ *and* $r_1, r_2 \in \mathbb{R}$ *with* $d(x_1, x_2) > r_1 + r_2$ *we can compute some* $n \in \mathbb{N}$ *such that* f *oscillates at most* n *times between* $B_1(x_1, r_1)$ *and* $B_2(x_2, r_2)$.

Proof. For some sufficiently large $n \in \mathbb{N}$, there must exist $t_1^1 < t_1^2 < t_2^1 \ldots < t_n^1 < t_n^2 \in S^1$ such that $f([t_i^1, t_i^2]) \subseteq \overline{B}(x_1, r_1)^C$, $f([t_i^2, t_{i+1}^2]) \subseteq \overline{B}(x_1, r_1)^C$ (with modular arithmetic on the indices). By overtness of S^1 and the availability of the compact-open representation on $\mathcal{C}(S^1, [0,1]^\omega)$, this is recognizable. Any such n is a valid answer, so we can just search for one that works.

Lemma 4. $\lim \leq_W \text{Image}^{-1}$.

Proof. We actually show that $\widehat{\text{UpperBound}} \leq \text{Image}^{-1}$ for $\text{UpperBound} :\subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathbb{N}$. The set we produce initially is a canonic unit circle, and we can provide its name in $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$ as a sequence of "tubes" shrinking with some known speed to it. For each of the ω-many UpperBound-instances we need to handle, we assign two open balls intersecting the unit circle, such that all balls are disjoint and the two balls for the same instance are next to each other. The situation at the site for a single UpperBound-instance is depicted in Figure 1.

Once a new, larger value n gets enumerated into an UpperBound-instance we modify our current circle in a way that forces any function spanning it to oscillate at least n times between the two balls assigned to that instance as shown in Figure 1.

For each site, we only need to do finitely many updates. This ensures that we actually end up with a topological circle. From any function $f : S^1 \to [0,1]^\omega$ having our circle as an image, we can extract upper bounds for how often it is oscillating at each side by Lemma 3, and these bounds are valid answers to the UpperBound-instances.

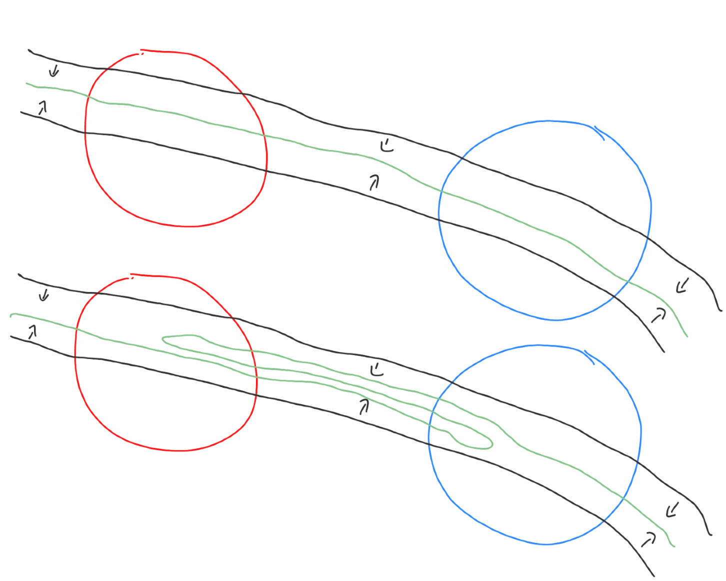

Fig. 1. The initial stage of the construction for Lemma 4 is shown at the top, the first update is depicted below. The tube going across the picture is the current approximation (which keeps shrinking). The actual circle we have in mind is depicted by the line lying in the tube and the two disjoint balls corresponding to one UpperBound-instance are depicted lying across the tube.

Recall that a *circular chain* is a sequence $B_0, \dots, B_n$ of open balls such that B_i and B_{i+1} intersect, as do B_0 and B_n, and any other pair of balls is formally disjoint (i.e. the distance between their centres is greater than the sum of their radii). We will say that a circular chain $C_0, \dots, C_m$ *refines* the circular chain $B_0, \dots, B_n$ *without backtracking* if there is a partition $\{0, \dots, m\} = \bigcup_{j \leq n} I_j$ into **intervals** such that $\forall i \in I_j\ \overline{C}_i \subseteq B_j$. A circular chain $B_0, \dots, B_n$ *strictly covers* a set S if $S \subseteq \bigcup_{i \leq n} B_n$ and $\forall i \leq n\ S \cap B_n \neq \emptyset$. A circular chain has scale ε if each radius of a ball occurring in it is smaller than ε.

Lemma 5. *Let S be a topological circle. From a sequence of circular chains $(\mathfrak{B}^i)_{i \in \mathbb{N}}$ all strictly covering S such that $\mathfrak{B}^{i+1}$ has scale 2^{-i} and refines $\mathfrak{B}^i$ without backtracking, we can compute a homeomorphism $f : S^1 \to S$.*

Lemma 6. *Let S be a topological circle and consider a circular chain $(B_0, \dots, B_n)$ strictly covering S. Either there are circular chains strictly covering S and refining $(B_0, \dots, B_n)$ of arbitrarily small positive scale, or there exists a circular chain $(C_0, \dots, C_m)$ strictly covering S such that $\forall j \leq m\ \exists i \leq n \overline{C}_j \subseteq B_i$ such that there are $j_0 < j_1 < j_2 < j_3$ and i with $\overline{C}_{j_0}, \overline{C}_{j_2} \subseteq B_i \setminus \overline{B}_{i+1}^C$ and $\overline{C}_{j_1}, \overline{C}_{j_3} \subseteq B_{i+1} \setminus \overline{B}_i^C$.*

Proof (Proof of Theorem 4). The reduction from (1) to (2) is the statement of Lemma 4. The reduction from (2) to (3) is trivial. To see that lim suffices to actually compute a homeomorphism, we consider the tree of circular chains strictly covering our circle where all balls have rational centres and radii ordered by refinement without backtracking on half the scale. By Lemma 5, having an infinite path through this tree suffices to find the desired homeomorphism. The basic parent-child relation in our tree is computably enumerable, and moreover, by invoking Lemma 6 we can recognize all dead ends. Thus, lim suffices to compute an infinite path.

4 Genericity

For $a \in A$, let $X_a = \mathrm{El}^{-1}(\{a\})$. We say that $a, b \in A$ are *homeomorphic* if $X_a \simeq X_b$. Morally, homeomorphic elements are names for the same object.

We call $U \subseteq A$ closed under homeo(morphism) if for all $a \in U$ and $b \in A$, if $X_b \simeq X_a$ then $b \in U$. We call $U \subseteq A$ *dense up to homeo(morphism)* if it intersects all open subsets of A which are closed under homeo.

Definition 6. *Let* $\mathrm{El}_A \colon A_\bullet \to A$ *be a bundle. Let* $\mathcal{C}$ *be a class of sets.*

Consider a space X *with* $X \simeq X_a$ *for some* $a \in A$*. The space* X *is called* $\mathcal{C}$*-generic as an element of* A *if for all* $C \subseteq A$ *belonging to* $\mathcal{C}$ *which are closed under homeomorphism and dense up to homeomorphism, we have* $a \in C$*.*

Morally, in the definition of genericity, we consider the represented space of homeomorphism types over a given space of spaces. A generic space is one whose homeomorphism type is contained in every dense set of class $\mathcal{C}$ in the space of homeomorphism types. This definition ensures that intensionally equivalent bundles have the same generic spaces.

We call a class $\mathcal{C}$ of sets a *pointclass* if it has the following properties:

1. The class $\mathcal{C}$ is closed under preimages of continuous maps.
2. If $e \colon \widetilde{A} \to A$ is a quotient map, then a set $C \subseteq A$ belongs to $\mathcal{C}$ if and only if $C = e\left(\widetilde{C}\right)$ for some $\widetilde{C} \in \mathcal{C}$ with $\widetilde{C} = e^{-1}\left(e\left(\widetilde{C}\right)\right)$.

Thus, a subset C of a represented space X is a member of a pointclass $\mathcal{C}$ if and only if it is the image of a set of names $\widetilde{C} \in \mathcal{C}$ under the representation, where $\widetilde{C}$ is extensional in the sense that a name of $x \in X$ belongs to $\widetilde{C}$ if and only if all names of x belong to $\widetilde{C}$. For a motivation of this definition, see [8], [10], [16].

Proposition 2. *Let* $\mathcal{C}$ *be a pointclass. Let* $\mathrm{El}_A \colon A_\bullet \to A$ *and* $\mathrm{El}_B \colon B_\bullet \to B$ *be equivalent bundles. Then a space* X *is* $\mathcal{C}$*-generic as an element of* A *if and only if it is* $\mathcal{C}$*-generic as an element of* B*.*

Proof. Assume that X is $\mathcal{C}$-generic as an element of A. We show that X is $\mathcal{C}$-generic as an element of B. By symmetry, this establishes the claim.

By assumption we have a diagram

$$\begin{array}{ccccc} A_\bullet & \xrightarrow{f_\bullet} & B_\bullet & \xrightarrow{g_\bullet} & A_\bullet \\ \downarrow{\scriptstyle \mathrm{El}_A} & & \downarrow{\scriptstyle \mathrm{El}_B} & & \downarrow{\scriptstyle \mathrm{El}_A} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & A \end{array}$$

where all the squares are pullbacks.

Let $a \in A$ with $X \simeq X_a$. Let $\mathcal{C} \ni C \subseteq B$ be closed under homeo and dense up to homeo. We claim that $f^{-1}(C)$ is closed under homeo and dense up to homeo. Then $a \in f^{-1}(C)$ by assumption, so that $f(a) \in C$ with $X \simeq X_{f(a)}$.

Let a_0, a_1 be homeomorphic elements of A with $f(a_0) \in C$. Then $f(a_1)$ is homeomorphic to $f(a_0)$ so that $f(a_1) \in C$. Hence, $f^{-1}(C)$ is closed under homeo.

Now, let $U \in \mathcal{O}(A)$ be an open set which is closed under homeo. Then the set $g^{-1}(U)$ is closed under homeo. It hence intersects C. Let $b \in C \cap g^{-1}(U)$. Then $g(b) \in U$. We claim that $g(b) \in f^{-1}(C)$. Indeed, $X_{f \circ g(b)} \simeq X_b$, so that $f \circ g(b) \in C$ since C is closed under homeo.

Proposition 3. *Let $\mathcal{C}$ be a pointclass. Let $\mathrm{El}_A \colon A_\bullet \to A$ be a bundle. Let $\mathrm{El}_{\widetilde{A}} \colon \widetilde{A}_\bullet \to \widetilde{A}$ be a projective reindexing of A. Then a space X is $\mathcal{C}$-generic as an element of A if and only if it is $\mathcal{C}$-generic as an element of $\widetilde{A}$.*

Proof. Assume that X is $\mathcal{C}$-generic as an element of A. Pick $a \in A$ with $X \simeq X_a$. Pick some $\widetilde{a} \in \widetilde{A}$ with $e_A(\widetilde{a}) = a$. Then $X \simeq X_{\widetilde{a}}$. Let $\mathcal{C} \ni \widetilde{C} \subseteq \widetilde{A}$ be closed under homeo and dense up to homeo. Since $\mathcal{C}$ is closed under homeo it particularly satisfies $\mathcal{C} = e_A^{-1}(e_A(\mathcal{C}))$. Hence, since $\mathcal{C}$ is a pointclass, we have $e_A(\mathcal{C}) \in \mathcal{C}$. It is easy to see that $e_A(\mathcal{C})$ is closed under homeo and dense up to homeo. It follows that $e_A(\mathcal{C})$ contains a. Hence $\mathcal{C} = e_A^{-1}(e_A(\mathcal{C}))$ contains $\widetilde{a}$.

Conversely, assume that X is $\mathcal{C}$-generic as an element of $\widetilde{A}$. Pick $\widetilde{a} \in \widetilde{A}$ with $e_A(\widetilde{a}) = a$. Let C be closed under homeo and dense up to homeo. Then the set $e_A^{-1}(C)$ is closed under homeo and dense up to homeo. It hence contains $\widetilde{a}$, so that $C = e_A\left(e_A^{-1}(C)\right)$ contains a.

Corollary 3. *Let $\mathcal{C}$ be a pointclass. Let $\mathrm{El}_A \colon A_\bullet \to A$ and $\mathrm{El}_B \colon B_\bullet \to B$ be intensionally equivalent bundles. Then a space X is $\mathcal{C}$-generic as an element of A if and only if it is $\mathcal{C}$-generic as an element of B.*

We work with density up to homeo rather than density, because the former is better behaved under pullbacks. For sufficiently nice bundles, there is a close connection between density and density up to homeo. Say that $\mathrm{El} \colon A_\bullet \to A$ is an *open representation* if the map

$$[\cdot] \colon A \to \mathcal{V}(A),\ a \mapsto \{b \in A \mid X_a \simeq X_b\}$$

is computable.

Proposition 4. *Let $\mathrm{El}_A \colon A_\bullet \to A$ be an open representation. Let $X \subseteq A$ be closed under homeo. Then X is dense up to homeo if and only if it is dense.*

Proof. Let $U \in \mathcal{O}(A)$ be an open set. Since El_A is an open representation, the set

$$V = \{a \in A \mid [a] \cap U \neq \emptyset\}$$

is open. Since X is dense up to homeo, there exists $a \in V \cap X$. By construction, there exists $b \in U$ with $X_a \simeq X_b$. Since X is closed under homeo we have $b \in X$.

Our bundles for Polish spaces and compact Polish spaces are open representations. We illustrate this with two examples

Proposition 5. *The bundle* $\mathrm{PM}_\bullet \to \mathrm{PM}$ *is an open representation.*

Proof (Sketch). Basic open sets in PM yield constraints on a pseudometric of the form

$$d(i_1, j_1) \in [a_1, b_1] \wedge \cdots \wedge d(i_m, j_m) \in [a_m, b_m].$$

All these constraints reveal about a space up to homeomorphism is that the space has at least k distinct points for some $k \geq 1$.

Hence, we can compute the homeomorphism class of such a space as an overt set essentially as follows: given a basic open set that requires the space to have k distinct point, accept the set if and only if our space has at least k distinct points.

Proposition 6. *The bundle* $(\mathcal{V} \wedge \mathcal{K})\left([0,1]^\omega\right)_{+,\bullet} \to (\mathcal{V} \wedge \mathcal{K})\left([0,1]^\omega\right)_+$ *is an open representation.*

Proof (Sketch). Basic open sets in $(\mathcal{V} \wedge \mathcal{K})\left([0,1]^\omega\right)$ correspond to minimal covers of a set by balls with rational centres and radii. All these constraints reveal about a space up to homeomorphism is that the space has at least k connected components for some $k \geq 1$.

Hence, we can compute the homeomorphism class of a space as an overt set essentially as follows: given a basic open set that requires the space to have k connected components, accept the set if and only if our space has at least k connected components.

We now turn to the problem of classifying generic compact Polish spaces. In view of Corollary 3 and Theorem 2 we are justified in calling a space $\mathcal{C}$-generic as a compact Polish space if it is generic as an element of any of our intensionally equivalent representations of compact Polish spaces. It follows from the proof of Proposition 6 that $\mathcal{O}$-genericity is rather weak: a compact Polish space is $\mathcal{O}$-generic if and only if it has infinitely many connected components. We next show that Cantor space is up to homeomorphism the only Π^0_2-generic compact Polish space.

Proposition 7. *Being homeomorphic to Cantor space is a* Π^0_2*-property of compact Polish spaces that is closed under homeo and dense up to homeo.*

Proof. We encode compact Polish spaces by the bundle with base $(\mathcal{V} \wedge \mathcal{K})\left([0,1]^\omega\right)_+$.

Let $(x_n)_n$ be a computable dense sequence in $[0,1]^\omega$. We denote by $B(x,r) \subseteq [0,1]^\omega$ the open ball of radius r and by $\overline{B}(x,r)$ the corresponding closed ball. Observe that the closed ball is the closure of the open ball. Consider the following Π^0_2-property:

$$\begin{aligned}
&\forall n \in \mathbb{N}.\forall r \in \mathbb{Q}_{>0}.\exists m \in \mathbb{N}.\exists s \in \mathbb{Q}_{>0}.\exists m' \in \mathbb{N}.\exists s \in \mathbb{Q}_{>0}.\\
&\qquad \big(B(x_n,r) \cap X \neq \emptyset \rightarrow \big(\overline{B}(x_n,r) \cap X\big) \subseteq B(x_m,s) \cup B(x_{m'},s')\\
&\qquad\qquad\qquad\qquad\qquad\qquad \wedge B(x_m,s) \cap X \neq \emptyset\\
&\qquad\qquad\qquad\qquad\qquad\qquad \wedge B(x_{m'},s') \cap X \neq \emptyset\\
&\qquad\qquad\qquad\qquad\qquad\qquad \wedge \overline{B}(x_m,s) \cap \overline{B}(x_{m'},s') \cap X = \emptyset\big).
\end{aligned}$$

Less formally, this property is stating that for every non-empty open rational ball, the corresponding closed ball can be covered by two non-empty open balls such that the corresponding closed balls are disjoint. It is easy to see that a set X has the above property if and only if it is homeomorphic to Cantor space. That this property is closed under homeo is obvious. It is dense up to homeo, since every open set that is closed under homeo contains a copy of Cantor space.

Proposition 8. *Every Π^0_2-subset of $(\mathcal{V} \wedge \mathcal{K})\left([0,1]^\omega\right)_+$ that is closed under homeo and dense up to homeo contains a copy (and hence every copy) of Cantor space.*

Proof. Proposition 7 establishes that the set $C_{\{0,1\}^\mathbb{N}}$ of all spaces homeomorphic to Cantor space is a dense Π^0_2-subset of $(\mathcal{V} \wedge \mathcal{K})\left([0,1]^\omega\right)_+$. Let C be a Π^0_2-subset of $(\mathcal{V} \wedge \mathcal{K})\left([0,1]^\omega\right)_+$ which is dense up to homeo and closed under homeo. Then by Proposition 4, C is dense. By the Baire category theorem, $C \cap C_{\{0,1\}^\mathbb{N}}$ is dense. If C does not contain a copy of Cantor space, then $C \cap C_{\{0,1\}^\mathbb{N}} = \emptyset$, contradiction!

Propositions 7 and 8 together establish:

Theorem 5. *Cantor space is, up to homeomorphism, the unique Π^0_2-generic compact Polish space.*

5 Further questions

Further classes of represented spaces where finding suitable representations as bundles would be of immense interest include the QuasiPolish and the coPolish spaces. For the former, much of the groundwork was laid in [6], where it was shown that we can represent QuasiPolish spaces as spaces of ideals of preorders. Subsequent work by de Brecht ([3], [2], [4]) showed that many of the constructions we would want to perform on QuasiPolish spaces are indeed effective using this encoding.

The setting for coPolish spaces is less well understood. In particular, it is not even completely clear what a "computable coPolish space" should be. The primary objective here is to check to what extend the various characterizations (such as those provided in [18]) of coPolish spaces still work in the effective setting. The investigation needed here will require a representation of the QuasiPolish spaces, as we would want the representation of a coPolish space $\mathbf{X}$ to enable us to compute the QuasiPolish space $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ for any given QuasiPolish space $\mathbf{Y}$.

Brattka discusses how to define computability of separable and some non-separable Banach spaces in [1], with a key insight being that if we represent separable Banach spaces as Polish spaces with additional structure, their duals naturally carry a coPolish topology. By expressing this in our framework, we would then have a general framework to study Banach spaces and their genericity, computable categoricity and so on.

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Appendix: proof of Lemma 1

Before moving on to the proof of Lemma 1, we first explain how the notions of Section 2.2 are actual general categorical notions. Then we use that to prove Lemma 1 via abstract nonsense. We hope this demonstrates in particular that the approach we use for represented spaces of represented spaces could be also done mutatis mutandi for other similar settings such as multirepresented spaces of multirepresented spaces.

Definition 7. *In a category, a* regular epimorphism *is a coequalizer of some pair of morphisms.*

In less categorical terms, regular epimorphisms are those surjections that arise from quotient maps. A characterization of regular epimorphisms in (multi)-represented spaces is that they have a multi-valued inverse; this is what we called *quotient maps* in Definition 2.

Example 3. The injections $2 \rightarrowtail \mathbb{S}$ are epimorphic but not regular.

Definition 8. *An object X of a category is called* (regular) projective *when it has the left lifting property against regular epimorphisms, that is, whenever we have a regular epimorphism $e : Z \twoheadrightarrow Y$ and a map $f : X \to Y$, there exists some $g : X \to Z$ such that $f = e \circ g$.*

$$\begin{tikzcd} & Z \arrow[d, two heads, "e"] \\ X \arrow[ur, dotted] \arrow[r, "f"'] & Y \end{tikzcd}$$

A (regular) projective cover *of an object X is a projective object $\widetilde{X}$ together with a regular epimorphism $\widetilde{X} \twoheadrightarrow X$*

In represented spaces, regular projective objects coincide with subspaces of $\mathbb{N}^{\mathbb{N}}$. It is also the case that a representation map $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \twoheadrightarrow X$ determines a regular projective cover of the space X by its domain. Hence Definition 2 says that a projective reindexing is really a pullback along a regular projective cover.

Remark 1. In represented spaces, for any projective cover $c : \widetilde{X} \twoheadrightarrow X$ and multivalued function $f : X \rightrightarrows Y$, a computable realizer for f allows to build a computable map $\widetilde{f} : \widetilde{X} \to Y$ such that $\widetilde{f}(z) \in f(c(z))$ for every $z \in \widetilde{X}$.

Lemma 1. *Any two projective reindexings of a bundle are equivalent.*

Proof. Let $\mathrm{El}_A : A_\bullet \to A$ be a bundle and $\mathrm{El}_B, \mathrm{El}_C$ be projective reindexings, so we have regular epimorphisms e_B and e_C as in the following diagram.

$$\begin{tikzcd}
B_\bullet \arrow[r] \arrow[d, "\mathrm{El}_B"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_\bullet \arrow[d, "\mathrm{El}_A"'] & C_\bullet \arrow[l] \arrow[d, "\mathrm{El}_C"] \arrow[dl, phantom, "\llcorner", very near start] \\
B \arrow[r, two heads, "e_B"'] & A & C \arrow[l, two heads, "e_C"]
\end{tikzcd}$$

To conclude, it suffices to show that El_B embeds into El_C. Since B is regular projective and e_C is a regular epimorphism, there is a map $f : B \to C$ such that $e_C \circ f = e_B$. Then, using the fact that the right square is a pullback, we know there exists $f_\bullet$ such that the following diagram commute.

$$\begin{tikzcd}
B_\bullet \arrow[r] \arrow[d, "\mathrm{El}_B"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, dashed, "f_\bullet"] & A_\bullet \arrow[d, "\mathrm{El}_A"'] & C_\bullet \arrow[l] \arrow[d, "\mathrm{El}_C"] \arrow[dl, phantom, "\llcorner", very near start] \\
B \arrow[r, two heads, "e_B"'] \arrow[rr, bend right, dashed, "f"'] & A & C \arrow[l, two heads, "e_C"]
\end{tikzcd}$$

Then it remains to show that $f, f_\bullet, \mathrm{El}_B$ and El_C form a pullback square; this follows easily from the fact that the left square in the original diagram is a pullback.

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