On the Weihrauch degree of the additive Ramsey theorem over the rationals*

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Abstract. We characterize the strength, in terms of Weihrauch degrees, of certain problems related to Ramsey-like theorems concerning colourings of the rationals. The theorems we are chiefly interested in assert the existence of almost-homogeneous sets for colourings of pairs of rationals satisfying properties determined by some additional algebraic structure on the set of colours.
In the context of reverse mathematics, most of the principles we study are equivalent to Σ^0_2-induction over RCA_0. The associated problems in the Weihrauch lattice are related to $\mathsf{TC}^*_{\mathbb{N}}$, $(\mathsf{LPO}')^*$ or their product, depending on their precise formalizations.

Keywords: Weihrauch reducibility, Reverse mathematics, additive Ramsey, Σ^0_2-induction.

1 Introduction

The infinite Ramsey theorem is a central object of study in the field of computability theory. It says that for any colouring c of n-uples of a given arity of an infinite set X, there exists a infinite subset $H \subseteq X$ such that the set of n-tuples $[H]^n$ of elements of H is homogeneous. This statement is non-constructive: even if the colouring c is given by a computable function, it is not the case that we can find a computable homogeneous subset of X. Various attempts have been made to quantify how non-computable this problem and some of its natural restrictions are. This is in turn linked to the axiomatic strength of the corresponding theorems, as investigated in *reverse mathematics* [12] where Ramsey's theorem is a privileged object of study [7].

This paper is devoted to a variant of Ramsey's theorem with the following restrictions: we colour pairs of rational numbers and we require some additional structure on the colouring, namely that it is *additive*. A similar statement first appeared in [11, Theorem 1.3] to give a self-contained proof of decidablity of the Monadic Second-order logic of $(\mathbb{Q}, <)$. We will also analyse a simpler statement we call the *shuffle principle*, a related tool appearing in more modern decidability proofs [4, Lemma 16]. The shuffle principle states that every $\mathbb{Q}$-indexed word (with letters in a finite alphabet) contains a convex subword in which every letter appears densely or not at all. Much like the additive restriction of the Ramsey

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theorem for pairs over $\mathbb{N}$, studied from the point of view of reverse mathematics in [8], we obtain a neat correspondence with Σ^0_2-induction (Σ^0_2-IND).

Theorem 1. *In the weak second-order arithmetic* RCA_0*,* Σ^0_2-IND *is equivalent to both the shuffle principle and the additive Ramsey theorem for* $\mathbb{Q}$*.*

We take this analysis one step further in the framework of Weihrauch reducibility that allows to measure the uniform strength of general multi-valued functions (also called *problems*) over Baire space. Let $\mathsf{Shuffle}$ and $\mathsf{ART}_{\mathbb{Q}}$ be the most obvious problems corresponding to the shuffle principle and additive Ramsey theorem over $\mathbb{Q}$ respectively. We relate them, as well as various weakenings $\mathsf{cShuffle}$, $\mathsf{cART}_{\mathbb{Q}}$, $\mathsf{iShuffle}$ and $\mathsf{iART}_{\mathbb{Q}}$ that only output sets of colours or intervals, to the standard (incomparable) problems $\mathsf{TC}_{\mathbb{N}}$ and LPO'.

Theorem 2. *We have the following equivalences*

- $\mathsf{Shuffle} \equiv_{\mathrm{W}} \mathsf{ART}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}} \times (\mathsf{LPO}')^*$
- $\mathsf{cShuffle} \equiv_{\mathrm{W}} \mathsf{cART}_{\mathbb{Q}} \equiv_{\mathrm{W}} (\mathsf{LPO}')^*$
- $\mathsf{iShuffle} \equiv_{\mathrm{W}} \mathsf{iART}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}}$

2 Background

In this section, we will introduce the necessary background for the rest of the paper, and fix most of the notation that we will use, except for formal definitions related to weak subsystems of second-order arithmetic, in particular RCA_0 (which consists of Σ^0_1-induction and recursive comprehension) and $\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IND}$. A standard reference for that material and, more generally, systems of interest in reverse mathematics, is [12].

2.1 Generic notations

We identify $k \in \mathbb{N}$ with the finite set $\{0, \dots, k-1\}$. For every linear order $(X, <_X)$, we write $[X]^2$ for the set of pairs (x, y) with $x <_X y$. In this paper, by an *interval* I we always mean a pair $(u, v) \in [\mathbb{Q}]^2$, regarded as the set $]u, v[$ of rationals; we never use interval with irrational extrema.

2.2 Additive and ordered colourings

For the following definition, fix a linear order $(X, <_X)$. For every poset $(P, \prec_P)$, we call a colouring $c : [X]^2 \to P$ *ordered* if we have $c(x, y) \preceq_P c(x', y')$ when $x' \leq_X x <_X y \leq_X y'$. A colouring $c : [X]^2 \to S$ is called *additive* with respect to a semigroup structure $(S, \cdot)$ if we have $c(x, z) = c(x, y) \cdot c(y, z)$ whenever $x <_X y <_X z$. A subset $A \subseteq X$ is *dense in* X if for every $x, y \in A$ with $x <_X y$ there is $z \in A$ such that $x <_X z <_X y$. Given a colouring $c : [X]^n \to k$ and some interval $Y \subseteq X$, we say that Y is *c-densely homogeneous* if there exists a finite partition of Y into dense subsets D_i such that each $[D_i]^n$ is monochromatic (that is, $|c([D_i]^n)| \leq 1$). We will call those *c-shuffles* if c happens to be a colouring of $\mathbb{Q}$ (i.e. $X = \mathbb{Q}$ and $n = 1$). Finally, given a colouring $c : \mathbb{Q} \to k$, and given an interval $I \subseteq \mathbb{Q}$, we say that a colour $i < k$ *occurs densely in* I if the set of $x \in \mathbb{Q}$ such that $c(x) = i$ is dense in I.

Definition 1. *The following are statements of second-order arithmetic:*

- $\mathsf{ORT}_{\mathbb{Q}}$*: for every finite poset* $(P, \prec_P)$ *and ordered colouring* $c : [\mathbb{Q}]^2 \to P$*, there exists a c-homogeneous interval* $]u, v[\subset \mathbb{Q}$*.*
- $\mathsf{Shuffle}$*: for every* $k \in \mathbb{N}$ *and colouring* $c : \mathbb{Q} \to k$*, there exists an interval* $I =]x, y[$ *such that* I *is a c-shuffle.*
- $\mathsf{ART}_{\mathbb{Q}}$*: for every finite semigroup* $(S, \cdot)$ *and additive colouring* $c : [\mathbb{Q}]^2 \to S$*, there exists an interval* $I =]x, y[$ *such that* I *is c-densely homogeneous.*

As mentioned before, a result similar to $\mathsf{ART}_{\mathbb{Q}}$ was originally proved by Shelah in [11, Theorem 1.3 & Conclusion 1.4] and $\mathsf{Shuffle}$ is a central lemma when analysing labellings of $\mathbb{Q}$ (see e.g. [4]). We will establish that $\mathsf{ART}_{\mathbb{Q}}$ and $\mathsf{Shuffle}$ are equivalent to Σ^0_2-induction over RCA_0 while $\mathsf{ORT}_{\mathbb{Q}}$ is provable in RCA_0.

We introduce some more terminology that will come in handy later on. Given a colouring $c : [\mathbb{Q}]^n \to k$, a set $C \subseteq k$ and an interval $I =]u, v[$ that is a c-shuffle, we say that I is a *c-shuffle for the colours in* C, or equivalently that I is *c-homogeneous for the colours of* C, if we additionally have $c(I) = C$.

2.3 Preliminaries on Weihrauch reducibility

We now give a brief introduction to the Weihrauch degrees of problems and the operations on them that we will use in the rest of the paper. We stress that here we are able to offer but a glimpse of this vast area of research, and we refer to [2] for more details on the topic.

We deal with partial multifunctions $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$, which we call *problems*, for short. We will most often define problems in terms of their *inputs* and of the *outputs* corresponding to those inputs. We stress that, differently from [2], we do not define problems for arbitrary represented spaces (domains and codomains of the problems we consider admit a straightforward coding as subspaces of $\mathbb{N}^{\mathbb{N}}$).

A partial function $F : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called a *realizer for* f, which we denote by $F \vdash f$, if, for every $x \in \mathrm{dom}(f)$, $F(x) \in f(x)$. Given two problems f and g, we say that g is *Weihrauch reducible* to f, and we write $g \leq_{\mathrm{W}} f$, if there are two computable functionals H and K such that $K\langle FH, \mathsf{id}\rangle$ is a realizer for g whenever F is a realizer for f. We define strong Weihrauch reducibility similarly: for every two problems f and g, we say that g *strongly Weihrauch reduces* to f, written $g \leq_{\mathrm{sW}} f$, if there are computable functionals H and K such that $KFH \vdash g$ whenever $F \vdash f$. We say that two problems f and g are (strongly) Weihrauch equivalent if both $f \leq_{\mathrm{W}} g$ and $g \leq_{\mathrm{W}} f$ (respectively $f \leq_{\mathrm{sW}} g$ and $g \leq_{\mathrm{sW}} f$). We write this $\equiv_{\mathrm{W}}$ (respectively $\equiv_{\mathrm{sW}}$).

There are a number of useful structural operations on problems, which respect the quotient to Weihrauch degrees, that we need to introduce. The first one is the *parallel product* $f \times g$, which has the power to solve an instance of f and and instance of g at the same time. The *finite parallelization* of a problem f, denoted f^*, has the power to solve an arbitrary number of instances of f, provided that number is given as part of the input. Finally, the *compositional product* of two problems f and g, denoted $f * g$, corresponds basically to the most complicated problem that can be obtained as a composition of f paired with the identity, a recursive function and g paired with identity (that last bit allows us to keep track of the initial input when applying f).

Now let us list some of the most important[1] problems that we are going to use in the rest of the paper.

- $\mathsf{C}_{\mathbb{N}} \colon \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ (*closed choice on* $\mathbb{N}$) is the problem that takes as input an enumeration e of a (strict) subset of $\mathbb{N}$ and such that, for every $n \in \mathbb{N}$, $n \in \mathsf{C}_{\mathbb{N}}(e)$ if and only if $n \notin \mathrm{ran}(e)$ (where $\mathrm{ran}(e)$ is the *range* of e).
- $\mathsf{TC}_{\mathbb{N}} \colon \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ (*totalization of closed choice on* $\mathbb{N}$) is the problem that takes as input an enumeration e of *any* subset of $\mathbb{N}$ (hence now we allow the possibility that $\mathrm{ran}(e) = \mathbb{N}$) and such that, for every $n \in \mathbb{N}$, $n \in \mathsf{TC}_{\mathbb{N}}(e)$ if and only if $n \notin \mathrm{ran}(e)$ or $\mathrm{ran}(e) = \mathbb{N}$.
- $\mathsf{LPO} \colon 2^{\mathbb{N}} \to \{0,1\}$ (*limited principle of omniscience*) takes as input any infinite binary string p and outputs 0 if and only if $p = 0^{\mathbb{N}}$.
- $\mathsf{LPO}' \colon \subseteq 2^{\mathbb{N}} \to \{0,1\}$: takes as input (a code for) an infinite sequence $\langle p_0, p_1, \dots \rangle$ of binary strings such that the function $p(i) = \lim_{s \to \infty} p_i(s)$ is defined for every $i \in \mathbb{N}$, and outputs $\mathsf{LPO}(p)$.

The definition of LPO' could have been obtained by composing the one of LPO and the definition of jump as given in [2]: we include it for convenience. Intuitively, LPO' corresponds to the power of answering a single binary Σ^0_2-question. In particular, LPO' is easily seen to be (strongly) Weihrauch equivalent to both $\mathsf{IsFinite}$ and $\mathsf{IsCofinite}$, the problems accepting as input an infinite binary string p and outputting 1 if p contains finitely (respectively, cofinitely) many 1s, and 0 otherwise. We will use this fact throughout the paper.

Another problem of combinatorial nature, introduced in [5], will prove to be very useful for the rest of the paper.

Definition 2. ECT *is the problem whose instances are pairs* $(n, f) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ *such that* $f \colon \mathbb{N} \to n$ *is a colouring of the natural numbers with* n *colours, and such that, for every instance* (n, f) *and* $b \in \mathbb{N}$, $b \in \mathsf{ECT}(n, f)$ *if and only if*

$$\forall x > b\ \exists y > x\ (f(x) = f(y)).$$

Namely, ECT is the problems that, upon being given a function f of the integers with finite range, outputs a b such that, after that b, the palette of colours used is constant (hence its name, which stands for *e*ventually *c*onstant palette *t*ail). We will refer to suitable bs as *bounds* for the function f.

A very important result concerning ECT and that we will use throughout the paper is its equivalence with $\mathsf{TC}^*_{\mathbb{N}}$.

[1] Whereas LPO and $\mathsf{C}_{\mathbb{N}}$ have been widely studied, $\mathsf{TC}_{\mathbb{N}}$ is somewhat less known (and does not appear in [2]): we refer to [9] for an account of its properties, and to [1] for a deeper study of some principles close to it.

Lemma 1 ([5, Theorem 9]). $\mathsf{ECT} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$

Another interesting result concerning ECT is the following: if we see it as a statement of second-order arithmetic (ECT can be seen as the principle asserting that for every colouring of the integers with finitely many colours there is a bound), then ECT and Σ_2^0-IND are equivalent over RCA_0 (actually, over RCA_0^*).

Lemma 2 ([5, Theorem 7]). *Over* RCA_0, ECT *and* Σ_2^0-IND *are equivalent.*

Hence, thanks to the results above, it is clear why $\mathsf{TC}_{\mathbb{N}}^*$ appears as a natural candidate to be a "translation" of Σ_2^0-IND in the Weihrauch degrees.

We end this section with two technical results about Weihrauch degrees. The first one asserts that the two main problems that we use as benchmarks in the sequel, namely $(\mathsf{LPO}')^*$ and $\mathsf{TC}_{\mathbb{N}}^*$, are incomparable in the Weihrauch lattice.

Lemma 3. $(\mathsf{LPO}')^*$ *and* $\mathsf{TC}_{\mathbb{N}}^*$ *are Weihrauch incomparable. Hence, we have that* $(\mathsf{LPO}')^*, \mathsf{TC}_{\mathbb{N}}^* <_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^*$.

The second result asserts that the sequential composition of $\mathsf{LPO}' \times \mathsf{TC}_{\mathbb{N}}$ after $\mathsf{C}_{\mathbb{N}}$ can actually be computed by the parallel product of LPO', $\mathsf{TC}_{\mathbb{N}}^n$ and $\mathsf{C}_{\mathbb{N}}$. As customary, for every problem P we write P^n to mean $\underbrace{\mathsf{P} \times \cdots \times \mathsf{P}}_{n \text{ times}}$.

Lemma 4. *For every integers a and b and every problem* $\mathsf{P} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$, *it holds that* $((\mathsf{LPO}')^a \times \mathsf{TC}_{\mathbb{N}}^b) * \mathsf{P} \leq_{\mathrm{W}} (\mathsf{LPO}')^a \times \mathsf{TC}_{\mathbb{N}}^b \times \mathsf{P}$.

2.4 Green theory

Green theory is concerned with analysing the structure of ideals of finite semigroups, be they one-sided on the left or right or even two-sided. This gives rise to a rich structure to otherwise rather inscrutable algebraic properties of finite semigroups. We will need only a few related results, all of them relying on the definition of the *Green preorders* and of idempotents (recall that an element s of a semigroup is idempotent when $ss = s$).

Definition 3. *For a semigroup $(S, \cdot)$, define the* Green preorders *as follows:*

- $s \leq_{\mathcal{R}} t$ *if and only if* $s = t$ *or* $s \in tS = \{ta : a \in S\}$ *(suffix order)*
- $s \leq_{\mathcal{L}} t$ *if and only if* $s = t$ *or* $s \in St = \{at : a \in S\}$ *(prefix order)*
- $s \leq_{\mathcal{H}} t$ *if and only if* $s \leq_{\mathcal{R}} t$ *and* $s \leq_{\mathcal{L}} t$
- $s \leq_{\mathcal{J}} t$ *if and only if* $s \leq_{\mathcal{R}} t$ *or* $s \leq_{\mathcal{L}} t$ *or* $s \in StS = \{atb : (a, b) \in S^2\}$ *(infix order)*

The associated equivalence relations are written $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{J}$; their equivalence classes are called respectively $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, and $\mathcal{J}$-classes.

We conclude this section reporting, without proof, the two technical lemmas that will be needed in Section 4. Although not proved in second-order arithmetic originally, it is clear that their proofs goes through in RCA_0: besides straightforward algebraic manipulations, they only rely on the existence, for each finite semigroup $(S, \cdot)$, of an index $n \in \mathbb{N}$ such that s^n is idempotent for any $s \in S$.

Lemma 5 ([10, Proposition A.2.4]). *If $(S, \cdot)$ is a finite semigroup, $H \subseteq S$ an $\mathcal{H}$-class, and some $a, b \in H$ satisfy $a \cdot b \in H$ then for some $e \in H$ we know that $(H, \cdot, e)$ is a group.*

Lemma 6 ([10, Corollary A.2.6]). *For any pair of elements $x, y \in S$ of a finite semigroup, if we have $x \leq_{\mathcal{R}} y$ and x, y $\mathcal{J}$-equivalent, then x and y are also $\mathcal{R}$-equivalent.*

3 The shuffle principle and related problems

3.1 The shuffle principle in reverse mathematics

We start by giving a proof[2] of the shuffle principle in $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IND}$, since, in a way, it gives a clearer picture of some properties of shuffles that we use in the rest of the paper.

Lemma 7. $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IND} \vdash \mathsf{Shuffle}$

Proof. Let $c : \mathbb{Q} \to n$ be a colouring of the rationals with n colours. For any natural number k, consider the following Σ^0_2 formula $\varphi(k)$: "there exists a finite set $L \subseteq n$ of cardinality k and there exist $u, v \in \mathbb{Q}$ with $u < v$ such that $c(w) \in L$ for every $w \in \,]u, v[$". Since $\varphi(n)$ is true, it follows from the Σ^0_2 minimization principle that there exists a minimal k such that $\varphi(k)$ holds. Consider $u, v \in \mathbb{Q}$ and the set of colours L corresponding to this minimal k. We now only need to show that $]u, v[$ is a c-shuffle to conclude.

Let $a = c(x)$ for some $x \in \,]u, v[$. We need to prove that a occurs densely in $]u, v[$. Consider arbitrary $x, y \in \,]u, v[$ with $x < y$. We are done if we show that there exists some $w \in \,]x, y[$ with $c(w) = a$. So, suppose that there is no such w. By bounded Σ^0_1-comprehension, there exists a finite set $L' \subset n$ consisting of exactly those $b \in n$ which occur as values of $c\big|_{]x,y[}$. Clearly, $\varphi(|L'|)$ holds. However, $L' \subseteq L$, and by assumption $a \notin L'$, so $|L'| < k$, contradicting the choice of k as the minimal number such that $\varphi(k)$ holds. □

The proof above shows an important feature of shuffles: given a certain interval $]u, v[$, any of its subintervals having the fewest colours is a shuffle. Interestingly, the above implication reverses, so we have the following equivalence.

Theorem 3. *Over* RCA_0, $\mathsf{Shuffle}$ *is equivalent to* $\Sigma^0_2\text{-}\mathsf{IND}$.

We do not offer a proof of the reversal here; such a proof can easily be done by taking inspiration from the argument we give for Lemma 11. With this equivalence in mind, we now introduce Weihrauch problems corresponding to $\mathsf{Shuffle}$, beginning with the stronger one.

[2] From Leszek A. Kołodziejczyk, personal communication.

Definition 4. *We regard* $\mathsf{Shuffle}$ *as the problem with instances* $(k, c) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ *such that* $c : \mathbb{Q} \to k$ *is a colouring of the rationals with* k *colours, and such that, for every instance* (k, c)*, for every pair* $(u, v) \in [\mathbb{Q}]^2$ *and for every* $C \subseteq k$*,* $(u, v, C) \in \mathsf{Shuffle}(k, c)$ *if and only if* $]u, v[$ *is a c-shuffle for the colours in* C*.*

Note that the output of $\mathsf{Shuffle}$ contains two components that cannot be easily computed from one another. It is thus natural to define two weakenings that we also study here.

Definition 5. $\mathsf{iShuffle}$ *("i" for "interval") is the same problem as* $\mathsf{Shuffle}$ *save for the fact that a valid output only contains the interval* $]u, v[$ *which is a c-shuffle. Complementarily,* $\mathsf{cShuffle}$ *("c" for "colour") is the problem that only outputs a possible set of colours taken by a c-shuffle.*

We will first start analysing the weaker problems $\mathsf{cShuffle}$ and $\mathsf{iShuffle}$ and show they are respectively equivalent to $(\mathsf{LPO}')^*$ and $\mathsf{TC}_{\mathbb{N}}^*$. This will also imply that $\mathsf{Shuffle}$ is stronger than $(\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^*$, but the converse will require an entirely distinct proof.

3.2 Weihrauch complexity of the weaker shuffle problems

We first start by discussing $\mathsf{cShuffle}$ briefly. Showing that it is stronger than $(\mathsf{LPO}')^*$ is relatively straightforward.

Lemma 8. $(\mathsf{LPO}')^* \leq_{\mathrm{W}} \mathsf{cShuffle}$

Proof idea. By noting that $\mathsf{cShuffle}^2 \leq_{\mathrm{W}} \mathsf{cShuffle}$ by considering pairing of distinct colourings, it suffices to show $\mathsf{LPO}' \leq_{\mathrm{W}} \mathsf{cShuffle}$. The reduction is then obtained by computing, from the input of LPO', a map $f : \mathbb{Q} \to \mathbb{N}$ such that infinite sets are taken to dense sets by f^{-1}. □

The reversal is more difficult; in this case, it is helpful to be more precise, and give a better estimate of the number of instances of LPO' necessary to solve an instance (n, c) of $\mathsf{cShuffle}$.

Lemma 9. *Let* $\mathsf{cShuffle}_n$ *be the restriction of* $\mathsf{cShuffle}$ *to the instances of the form* (n, c)*. Then,* $\mathsf{cShuffle}_n \leq_{\mathrm{W}} (\mathsf{LPO}')^{2^n - 1}$

Proof idea. We use one instance of LPO' for each non-empty subset C of n, to decide if there is an interval in which only colours from C appear. The $\subseteq$-minimal C for which it happens are guaranteed to correspond to a c-shuffle. □

Putting the two previous results together, we have the following.

Theorem 4. $(\mathsf{LPO}')^* \equiv_{\mathrm{W}} \mathsf{cShuffle}$

Now we move to $\mathsf{iShuffle}$.

Lemma 10. *Let* $\mathsf{iShuffle}_n$ *be the restriction of* $\mathsf{iShuffle}$ *to the instances of the form* (n, c)*. For every* $n \in \mathbb{N}$ *with* $n \geq 2$*,* $\mathsf{iShuffle}_n \leq_{\mathrm{sW}} \mathsf{TC}_{\mathbb{N}}^{n-1}$*.*

Proof idea. Fix an enumeration of the intervals of $\mathbb{Q}$ and let (n, c) be an instance of $\mathsf{iShuffle}_n$. The idea of the reduction is the following. With the first instance e_{n-1} of $\mathsf{TC}_\mathbb{N}$, we look for an interval I on which c takes only $n-1$ colours: if no such interval exists, then this means that every colour is dense in every interval, and so every inverval would be a valid solution to c. Hence, we can suppose that such an interval is eventually found: we will then use the second instance e_{n-2} of $\mathsf{TC}_\mathbb{N}$ to look for a subinterval of I_j where c takes only $n-2$ values. Again, we can suppose that such an interval is found. We proceed like this for $n-1$ steps, so that in the end the last instance e_1 of $\mathsf{TC}_\mathbb{N}$ is used to find an interval I' inside an interval I on which we know that at most two colours appear. Again, we look for c-monochromatic intervals: if we do not find any, then I' is already a c-shuffle, whereas if we do find one, then that interval is a solution.

Although not apparent in the sketch given above, an important part of the proof is that the $n-1$ searches we described can be performed *in parallel*: the fact that this can be accomplished relies on the fact that any subinterval of a shuffle is a shuffle. □

Lemma 11. *Let* ECT_n *be the restriction of* ECT *to the instances of the form* (n, f)*. For every* $n \in \mathbb{N}$ *with* $n \geq 2$*,* $\mathsf{ECT}_n \leq_{\mathrm{sW}} \mathsf{iShuffle}_n$*.*

Proof. Let (n, f) be an instance of ECT_n. We will slightly abuse notation, in the following sense: we will define a colouring $c : \mathbb{D} \to n$ of the dyadics, instead of directly defining a colouring of the rationals. We will then exploit the fact that there is a computable order-preserving bijection between the dyadic numbers $\mathbb{D}$ and $\mathbb{Q}$, and we will apply $\mathsf{iShuffle}_n$ to (n, c).

We define $c\colon \mathbb{D} \to n$ as follows: let $d = \frac{m}{2^h}$ be a dyadic number, then we let $c(d) = f(h)$. Hence, all the points of the same denominator have the same colour according to c. Let $(\frac{u}{2^k}, \frac{v}{2^\ell}) \in \mathsf{iShuffle}_n(n, c)$. Let b be such that $\frac{1}{2^b} < \frac{v}{2^\ell} - \frac{u}{2^k}$. We claim that b is a bound for f. Suppose not, then there is a colour $i < n$ and a number $x \in \mathbb{N}$ such that $x > b$ and $f(x) = i$, but for no $y > x$ it holds that $f(y) = i$. Hence, all the dyadics of the form $\frac{w}{2^x}$ are given colour i, but i does not appear densely often in any interval of $\mathbb{D}$. But by definition of b, there is a $z \in \mathbb{N}$ such that $\frac{z}{2^x} \in \left]\frac{u}{2^k}, \frac{v}{2^\ell}\right[$, which is a contradiction. Hence b is a bound for f. □

We can then relate this to $\mathsf{TC}_\mathbb{N}$; the next lemma follows directly by inspecting the second half of [5, Theorem 9].

Lemma 12. *For every* $n \in \mathbb{N}$ *with* $n \geq 2$*,* $\mathsf{TC}_\mathbb{N}^{n-1} \leq_{\mathrm{W}} \mathsf{ECT}_n$ *(and this cannot be improved to a strong Weihrauch reduction).*

Putting things together, we finally have a characterization of $\mathsf{iShuffle}$.

Theorem 5. *For every* $n \geq 2$*, we have the Weihrauch equivalence*

$$\mathsf{ECT}_n \equiv_{\mathrm{W}} \mathsf{iShuffle}_n \equiv_{\mathrm{W}} \mathsf{TC}_\mathbb{N}^{n-1} \qquad \textit{whence} \qquad \mathsf{ECT} \equiv_{\mathrm{W}} \mathsf{iShuffle} \equiv_{\mathrm{W}} \mathsf{TC}_\mathbb{N}^*$$

3.3 The full shuffle problem

The main result of this section is that $\mathsf{Shuffle} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^* \times (\mathsf{LPO}')^*$, which will be proved in Theorem 6. In order to do that, it is convenient to observe that, similarly to $\mathsf{cShuffle}$ and $\mathsf{iShuffle}$, $\mathsf{Shuffle}$ is closed under finite parallelization.

Lemma 13. $\mathsf{Shuffle} \times \mathsf{Shuffle} \leq_{\mathrm{W}} \mathsf{Shuffle}$. *Therefore,* $\mathsf{Shuffle}^* \equiv_{\mathrm{W}} \mathsf{Shuffle}$.

This enables one to easily prove the following lemma.

Lemma 14. $\mathsf{TC}_{\mathbb{N}}^* \times (\mathsf{LPO}')^* \leq_{\mathrm{W}} \mathsf{Shuffle}$

Proof. From Theorem 4 and Theorem 5, we have that $\mathsf{TC}_{\mathbb{N}}^* \times (\mathsf{LPO}')^* \leq_{\mathrm{W}} \mathsf{iShuffle} \times \mathsf{cShuffle}$, and since clearly $\mathsf{iShuffle} \sqcup \mathsf{cShuffle} \leq_{\mathrm{W}} \mathsf{Shuffle}$, by Lemma 13 we have that $\mathsf{TC}_{\mathbb{N}}^* \times (\mathsf{LPO}')^* \leq_{\mathrm{W}} \mathsf{Shuffle}$. ☐

For the other direction, again, we want to be precise as to the number of instances of $\mathsf{TC}_{\mathbb{N}} \times (\mathsf{LPO}')$ needed to solve an instance of $\mathsf{Shuffle}$.

Lemma 15. *Let* $\mathsf{Shuffle}_n$ *be the restriction of* $\mathsf{Shuffle}$ *to the instances of the form* (n, c)*. Then,* $\mathsf{Shuffle}_n \leq_{\mathrm{W}} (\mathsf{TC}_{\mathbb{N}} \times \mathsf{LPO}')^{2^n-1}$

Proof idea. Let (n, c) be an instance of $\mathsf{Shuffle}$. Essentially, the main idea for the proof of $\mathsf{Shuffle}_n \leq_{\mathrm{W}} (\mathsf{TC}_{\mathbb{N}} \times \mathsf{LPO}')^{2^n-1}$ is to combine the proofs of Lemma 10 and of Theorem 4: we want to use $\mathsf{TC}_{\mathbb{N}}$ to find a candidate interval for a certain subset C of n, and on the side we use LPO' (or equivalently, $\mathsf{IsFinite}$) to check for every such set C whether a c-shuffle for the colours of C actually exists. The main difficulty with the idea described above is that the two proofs must be intertwined, in order to be able to find both a c-shuffle and the set of colours that appear on it. ☐

Putting the previous results together, we obtain the following.

Theorem 6. $\mathsf{Shuffle} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^* \times (\mathsf{LPO}')^*$

4 $\mathsf{ART}_{\mathbb{Q}}$ and related problems

We now analyse the logical strength of the principle $\mathsf{ART}_{\mathbb{Q}}$. As in the case of $\mathsf{Shuffle}$, we start with a proof of $\mathsf{ART}_{\mathbb{Q}}$ in $\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IND}$. This will give us enough insights to assess the strength of the corresponding Weihrauch problems.

4.1 Additive Ramsey over $\mathbb{Q}$ in reverse mathematics

As a preliminary step, we figure out the strength of $\mathsf{ORT}_{\mathbb{Q}}$, the ordered Ramsey theorem over $\mathbb{Q}$. It is readily provable from RCA_0 and is thus much weaker than most other principles we analyse. We can be a bit more precise by considering RCA_0^* which is basically the weakening of RCA_0 where induction is restricted to Δ^0_1 formulas (see [12, Definition X.4.1] for a nice formal definition).

Lemma 16. $\mathsf{RCA}_0^* \vdash \mathsf{RCA}_0 \Leftrightarrow \mathsf{ORT}_{\mathbb{Q}}$

We now show that the shuffle principle is equivalent to $\mathsf{ART}_{\mathbb{Q}}$. So overall, much like the Ramsey-like theorems of [8], they are equivalent to Σ^0_2-induction.

Lemma 17. $\mathsf{RCA}_0 + \mathsf{Shuffle} \vdash \mathsf{ART}_{\mathbb{Q}}$. *Hence,* $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IND} \vdash \mathsf{ART}_{\mathbb{Q}}$.

Proof. Fix a finite semigroup $(S, \cdot)$ and an additive colouring $c : [\mathbb{Q}]^2 \to S$. Say a colour $s \in S$ *occurs* in $X \subseteq \mathbb{Q}$ if there exists $(x, y) \in [X]^2$ such that $c(x, y) = s$.

We proceed in two stages: first, we find an interval $]u, v[$ such that all colours occurring in $]u, v[$ are $\mathcal{J}$-equivalent to one another. Then we find a subinterval of $]u, v[$ partitioned into finitely many dense homogeneous sets. For the first step, we apply the following lemma to obtain a subinterval $I_1 =]u, v[$ of $\mathbb{Q}$ where all colours lie in a single $\mathcal{J}$-class.

Lemma 18. *For every additive colouring c, there exists $(u, v) \in [\mathbb{Q}]^2$ such that all colours of $c\big|_{]u,v[}$ are $\mathcal{J}$-equivalent to one another.*

Proof. If we post-compose c with a map taking a semigroup element to its $\mathcal{J}$-class, we get an ordered colouring. Applying $\mathsf{ORT}_{\mathbb{Q}}$ yields a suitable interval. ☐

Moving on to stage two of the proof, we want to look for a subinterval of I_1 partitioned into finitely many dense homogeneous sets. To this end, define a colouring $\gamma : I_1 \to S^2$ by setting $\gamma(z) = (c(u, z), c(z, v))$.

By $\mathsf{Shuffle}$, there exist $x, y \in I_1$ with $x < y$ such that $]x, y[$ is a γ-shuffle. For $l, r \in S$, define $H_{l,r} := \gamma^{-1}(\{(l, r)\}) \subseteq]x, y[$; note that this is a set by bounded recursive comprehension. Clearly, all $H_{l,r}$ are either empty or dense in $]x, y[$, and moreover $]x, y[= \bigcup_{l,r} H_{l,r}$. Since there are finitely many pairs (l, r), all we have to prove is that each non-empty $H_{l,r}$ is homogeneous for c.

Let $s = c(w, z)$ such that $w, z \in H_{l,r}$ with $w < z$. By additivity of c and the definition of $H_{l,r}$,

$$s \cdot r = c(w, z) \cdot c(z, v) = c(w, v) = r. \tag{1}$$

In particular $r \leq_{\mathcal{R}} s$. But we also have $r \mathrel{\mathcal{J}} s$, which gives $r \mathrel{\mathcal{R}} s$ by Lemma 6. This shows that all the colours occurring in $H_{l,r}$ are $\mathcal{R}$-equivalent to one another. A dual argument shows that they are all $\mathcal{L}$-equivalent, so they are all $\mathcal{H}$-equivalent. The assumptions of Lemma 5 are satisfied, so their $\mathcal{H}$-class is actually a group.

All that remains to be proved is that any colour s occurring in $H_{l,r}$ is actually the (necessarily unique) idempotent of this $\mathcal{H}$-class. Since $r \mathrel{\mathcal{R}} s$, there exists a such that $s = r \cdot a$. But then by (1), $s \cdot s = s \cdot r \cdot a = r \cdot a = s$, so s is necessarily the idempotent. Thus, all sets $H_{l,r}$ are homogeneous and we are done. ☐

We conclude this section by showing that the implication proved in the Lemma above reverses., thus giving the precise strength of $\mathsf{ART}_{\mathbb{Q}}$ over RCA_0.

Theorem 7. $\mathsf{RCA}_0 + \mathsf{ART}_{\mathbb{Q}} \vdash \mathsf{Shuffle}$. *Hence,* $\mathsf{RCA}_0 \vdash \mathsf{ART}_{\mathbb{Q}} \leftrightarrow \Sigma^0_2\text{-}\mathsf{IND}$.

Proof. Let $f\colon \mathbb{Q} \to n$ be a colouring of the rationals. Let $(S_n, \cdot)$ be the finite semigroup defined by $S_n = n$ and $a \cdot b = a$ for every $a, b \in S_n$. Define the colouring $c\colon [\mathbb{Q}]^2 \to S_n$ by setting $c(x, y) = f(x)$ for every $x, y \in \mathbb{Q}$. Since for every $x < y < z$, $c(x, z) = f(x) = c(x, y) \cdot c(y, z)$, c is additive. By additive Ramsey, there exists $]u, v[$ which is c-densely homogeneous and thus a f-shuffle. ☐

4.2 Weihrauch complexity of additive Ramsey

We now start the analysis of $\mathsf{ART}_\mathbb{Q}$ in the context of Weihrauch reducibility. We will mostly summarize results, relying on the intuitions we built up so far. First off, we determine the Weihrauch degree of the ordered Ramsey theorem over $\mathbb{Q}$.

Theorem 8. *Let $\mathsf{ORT}_\mathbb{Q}$ be the problem whose instances are ordered colourings $c : [\mathbb{Q}]^2 \to P$, for some finite poset $(P, \prec)$, and whose possible outputs on input c are intervals on which c is constant. We have that $\mathsf{ORT}_\mathbb{Q} \equiv_\mathrm{W} \mathsf{LPO}^*$.*

Proof idea. $\mathsf{LPO}^* \leq_\mathrm{sW} \mathsf{ORT}_\mathbb{Q}$: given n sequences $p_0, \dots, p_{n-1} \in 2^\mathbb{N}$, build a coloring $c : [\mathbb{Q}]^2 \to 2^n$ such that, for every $(x, y) \in [\mathbb{Q}]^2$ and $l \in \mathbb{N}$ such that $2^{-l-1} \leq y - x < 2^{-l}$, $i \in c(x, y)$ if and only if there is $k < l$ such that $p_i(k) = 1$. This is an ordered coloring, and the color associated to any homogeneous set gives answer to $\mathsf{LPO}(p_i)$.

$\mathsf{ORT}_\mathbb{Q} \leq_\mathrm{W} \mathsf{LPO}^*$: without loss of generality, assume that the input is a coloring $c : [\mathbb{Q}]^2 \to k$ where k is ordered as usual. There is a straightforward procedure that, taking an interval I and a color $i \in k$, checks if there exists a pair of $(x, y) \in [I]^2$ such that $c(x, y) < i$, and returns that pair if it exists (and otherwise does not terminate). Now run that procedure for $i = k - 1$ and some arbitrary interval I_{k-1}, and if it returns some (x, y), run it for $i = k - 2$ and the interval $]x, y[$, and so forth (note that we cannot drop below $i = 0$ since the coloring is ordered). Calling $(x_s, y_s)_{s\in\mathbb{N}}$ the sequence of pairs that are tested, define the sequences p_i for every $i < k$ by $p_i(s) = 1 \Leftrightarrow c(x_s, y_s) < i$. The largest i such that $\mathsf{LPO}(p_i) = 0$ will be the color of some monochromatic interval that can be determined by the first s such that $p_{i+1}(s) = 1$ (or is I_{k-1} if $i = k - 1$). ☐

Now let us discuss Weihrauch problems corresponding to $\mathsf{ART}_\mathbb{Q}$.

Definition 6. *Regard $\mathsf{ART}_\mathbb{Q}$ as the following Weihrauch problem: the instances are pairs (S, c) where S is a finite semigroup and $c : [\mathbb{Q}]^2 \to S$ is an additive colouring of $[\mathbb{Q}]^2$, and such that, for every $C \subseteq S$ and every interval I of $\mathbb{Q}$, $(I, C) \in \mathsf{ART}_\mathbb{Q}$ if and only if I is c-densely homogeneous for the colours of C.*

Similarly to what we did in Definition 5, we also introduce the problems $\mathsf{cART}_\mathbb{Q}$ and $\mathsf{iART}_\mathbb{Q}$ that only return the set of colours and the interval respectively.

We start by noticing that the proof of Theorem 7 can be readily adapted to show the following.

Lemma 19. – $\mathsf{cShuffle} \leq_\mathrm{sW} \mathsf{cART}_\mathbb{Q}$, *hence* $(\mathsf{LPO}')^* \leq_\mathrm{W} \mathsf{cART}_\mathbb{Q}$.

– $\mathsf{iShuffle} \leq_{\mathrm{sW}} \mathsf{iART}_{\mathbb{Q}}$, *hence* $\mathsf{TC}_{\mathbb{N}}^* \leq_{\mathrm{W}} \mathsf{iART}_{\mathbb{Q}}$.
– $\mathsf{Shuffle} \leq_{\mathrm{sW}} \mathsf{ART}_{\mathbb{Q}}$, *hence* $(\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^* \leq_{\mathrm{W}} \mathsf{ART}_{\mathbb{Q}}$.

The rest of the section is devoted to find upper bounds for $\mathsf{cART}_{\mathbb{Q}}$, $\mathsf{iART}_{\mathbb{Q}}$ and $\mathsf{ART}_{\mathbb{Q}}$. The first step to take is a careful analysis of the proof of Lemma 17. For an additive colouring $c \colon [\mathbb{Q}]^2 \to S$, the proof can be summarized as follows:

– we start with an application of $\mathsf{ORT}_{\mathbb{Q}}$ to find an interval $]u, v[$ such that all the colours of $c\big|_{]u,v[}$ are all $\mathcal{J}$-equivalent (Lemma 18).
– define the colouring $\gamma \colon \mathbb{Q} \to S^2$ and apply $\mathsf{Shuffle}$ to it, thus obtaining the interval $]x, y[$.
– the rest of the proof consists simply in showing that $]x, y[$ is a c-densely homogeneous interval.

Hence, from the uniform point of view, this shows that $\mathsf{ART}_{\mathbb{Q}}$ can be computed via a composition of $\mathsf{Shuffle}$ and $\mathsf{ORT}_{\mathbb{Q}}$. Whence the next theorem.

Theorem 9. – $\mathsf{cART}_{\mathbb{Q}} \leq_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{LPO}^*$, *therefore* $\mathsf{cART}_{\mathbb{Q}} \equiv_{\mathrm{W}} (\mathsf{LPO}')^*$.
– $\mathsf{iART}_{\mathbb{Q}} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^* \times \mathsf{LPO}^*$, *therefore* $\mathsf{iART}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$.
– $\mathsf{ART}_{\mathbb{Q}} \leq_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^* \times \mathsf{LPO}^*$, *therefore* $\mathsf{ART}_{\mathbb{Q}} \equiv_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^*$.

5 Conclusion and future work

We have analysed the strength of an additive Ramseyan theorem over the rationals from the point of view of reverse mathematics and found it to be equivalent to Σ_2^0-induction, and then refined that analysis to a Weihrauch equivalence with $\mathsf{TC}_{\mathbb{N}}^* \times (\mathsf{LPO}')^*$. We have also shown that the problem decomposes nicely: we get the distinct complexities $(\mathsf{LPO}')^*$ or $\mathsf{TC}_{\mathbb{N}}^*$ if we only require either the set of colours or the location of the homogeneous set respectively. The same holds true for another equally and arguably more fundamental shuffle principle.

For further work, we believe it should be straightforward to carry out a similar analysis for Ramsey theorem over $\mathbb{N}$ (known to be equivalent to Σ_2^0-induction in the context of reverse mathematics [8]). Related to $\mathbb{Q}$, there are also weaker combinatorial principles of interest to look at like $(\eta)^1_{<\infty}$ from [6]. More generally, it would be interesting to study standard mathematical theorems that are known to be equivalent to Σ_2^0-IND in reverse mathematics: this can be considered to contribute to the larger endevour of studying principles already analyzed in reverse mathematics in the framework of the Weihrauch degrees. In the particular case of Σ_2^0-IND, it can be interesting to see which degrees are necessary for such an analysis. We refer to [3] for more on this topic, and for a more comprehensive study of Ramsey's theorem in the Weihrauch degrees.

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A Proof of Lemma 3

Lemma 3. $(\mathsf{LPO}')^*$ *and* $\mathsf{TC}_{\mathbb{N}}^*$ *are Weihrauch incomparable. Hence, we have that* $(\mathsf{LPO}')^*, \mathsf{TC}_{\mathbb{N}}^* <_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^*$.

Proof. $\mathsf{TC}_{\mathbb{N}}^* \not\leq_{\mathrm{W}} (\mathsf{LPO}')^*$: to do this, we actually show the stronger result that $\mathsf{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} (\mathsf{LPO}')^*$. Suppose for a contradiction that a reduction exists, as witnessed by the computable functionals H and K: this means that, for every instance e of $\mathsf{C}_{\mathbb{N}}$, $H(e)$ is an instance of $(\mathsf{LPO}')^*$, and for every solution $\sigma \in (\mathsf{LPO}')^*(H(e))$, $K(e, \sigma)$ is a solution to e, i.e. $K(e, \sigma) \in \mathsf{C}_{\mathbb{N}}(e)$. We build an instance e of $\mathsf{C}_{\mathbb{N}}$ contradicting this.

We start by letting e enumerate the empty set. At a certain stage s, by definition of instances of $(\mathsf{LPO}')^*$, $H(e|_s)$ converges to a certain n, the number of applications of LPO' that are going to be used in the reduction. Hence, however we continue the construction of e, there are at most 2^n possible values for $(\mathsf{LPO}')^*(H(e))$, call them $\sigma_0, \dots, \sigma_{2^n-1}$. It is now simple to diagonalize against all of them, one at a time, as we now explain. We let e enumerate the empty set until, for some s_0 and i_0, $K(e|_{s_0}, \sigma_{i_0})$ converges to a certain m_0: notice that such an i_0 has to exist, by our assumption that H and K witness the reduction of $\mathsf{C}_\mathbb{N}$ to $(\mathsf{LPO})^*$. Then, we let e enumerate m_0 at stage s_0+1: this implies that σ_{i_0} cannot be a valid solution to $H(e)$, otherwise $K(e, \sigma_{i_0})$ would be a solution to e. We then keep letting e enumerating $\{m_0\}$ until, for certain s_1 and i_1, $K(e|_{s_1}, \sigma_{i_1})$ converges to m_1. We then let e enumerate $\{m_0, m_1\}$, and continue the construction in this fashion. After 2^n many steps, we will have diagonalized against all the σ_i, thus reaching the desired contradiction.

$(\mathsf{LPO}')^* \not\leq_\mathrm{W} \mathsf{TC}_\mathbb{N}^*$: we use the fact that $\mathsf{TC}_\mathbb{N}^* \equiv_\mathrm{W} \mathsf{ECT}$ (see [5]). We will show a stronger result, namely that $\mathsf{IsFinite}_\mathbb{S} \not\leq_\mathrm{W} \mathsf{ECT}$, where $\mathsf{IsFinite}_\mathbb{S} : 2^\mathbb{N} \to \mathbb{S}$ is the following problem, as defined in [9]: for every $p \in 2^\mathbb{N}$, $\mathsf{IsFinite}_\mathbb{S}(p) = \top$ if p contains only finitely many occurrences of 1 and $\mathsf{IsFinite}_\mathbb{S}(p) = \bot$ otherwise [3].

Notice that $\mathsf{IsFinite}_\mathbb{S} \leq_\mathrm{W} \mathsf{LPO}'$: given any string $p \in 2^\mathbb{N}$, we consider the instance $\langle p_0, p_1, \dots \rangle$ of LPO' defined as follows: for every i, p_i takes value 1 until (and if) the ith occurrence of 1 is found in p, after which point it takes value 0. Then, $\mathsf{LPO}'(\langle p_0.p_1 \dots \rangle) = 1$ if and only if $\mathsf{IsFinite}_\mathbb{S}(p) = \bot$. Hence, if we show that $\mathsf{IsFinite}_\mathbb{S} \not\leq_\mathrm{W} \mathsf{ECT}$, we have in particular that $(\mathsf{LPO}')^* \not\leq_\mathrm{W} \mathsf{TC}_\mathbb{N}^*$.

Suppose for a contradiction that a reduction exists and is witnessed by functionals H and K. We build an instance p of $\mathsf{IsFinite}_\mathbb{S}$ contradicting this.

Let us consider the colouring $H(0^\mathbb{N})$, and let $b_0 \in \mathsf{ECT}(H(0^\mathbb{N}))$ be a bound for it. Since $\mathsf{IsFinite}_\mathbb{S}(0^\mathbb{N}) = \top$, there is an n_0 such that the following two conditions hold: $K(0^{n_0}, b_0)$ converges and gives as output $\top$, and moreover, the partial colouring $H(0^{n_0})$ is such that, for every colour j showing up after b_0 in $H(0^\mathbb{N})$ (i.e., every colour in the constant palette of $H(0^\mathbb{N})$), there is an $m > b_0$ such that $H()^{n_0})(m) = j$. We then consider the colouring $H(0^{n_0}10^\mathbb{N})$, and a bound b_1 for it. Again by the fact that $\mathsf{IsFinite}_\mathbb{S}(0^{n_0}10^\mathbb{N}) = \top$, there is an n_1 satisfying the following two conditions: $K(0^{n_0}10^{n_1}, b_1)$ converges to $\top$ and moreover, the partial colouring $H(0^{n_0}10^{n_1})$ is such that, for every colour j in the constant palette of $H(0^{n_0}10^\mathbb{N})$, there are *two* numbers $m > m' > b_1$ such that $H(0^{n_0}10^{n_1})(m) = H(0^{n_0}10^{n_1})(m') = j$. We then move to consider the colouring $H(0^{n_0}10^{n_1}10^\mathbb{N})$. We iterate the procedure infinitely many times.

Let $p \in 2^\mathbb{N}$ be the infinite binary string obtained as the limit of the process described in the previous paragraph, and notice that $\mathsf{IsFinite}_\mathbb{S}(p) = \bot$.

Let us consider the colouring $H(p)$ and a bound $b \in \mathsf{ECT}(H(p))$ for this colouring. If there exists an i such that $b \leq b_i$, where b_i is a bound found in the construction of p, then b_i is also a valid bound for $H(p)$. But then, by the construction, $K(p, b_i) = \top$, which cannot happen.

Hence, every bound b for $H(p)$ is larger than every bound b_i found during the construction. But then, there is a $b' < b$ such that for infinitely many i, $b' = b_i$. We claim that b' is a valid bound for $H(p)$. Suppose not: then, there is a colour j of $H(p)$ that appears only finitely many times after b', say k times. But since at stage s of the construction of p we forced every colour appearing after b_s to occur at least $s+1$ times after the bound b_s, by the fact that b' is chosen

[3] $\mathbb{S}$ is the Sierpinski space $\{\top, \bot\}$, where $\top$ is coded by the binary strings containing at least one 1, and $\bot$ is coded by $0^\mathbb{N}$. $\mathsf{IsFinite}_\mathbb{S}$ is strictly weaker than $\mathsf{IsFinite}$

as bound infinitely many times we can find a stage where we have forced j to appear $k+1$ times after b', thus proving that b' is a valid bound. Then, as in the previous case, we have that $K(p, b') = \top$, yielding the desired contradiction. □

B Proof of Lemma 4

Lemma 4. *For every integers a and b and every problem $\mathsf{P} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$, it holds that $((\mathsf{LPO}')^a \times \mathsf{TC}_{\mathbb{N}}^b) * \mathsf{P} \leq_{\mathrm{W}} (\mathsf{LPO}')^a \times \mathsf{TC}_{\mathbb{N}}^b \times \mathsf{P}$.*

Proof. Clearly, it is sufficient to prove that $((\mathsf{LPO}')^a \times \mathsf{TC}_{\mathbb{N}}^b) * \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} (\mathsf{LPO}')^a \times \mathsf{TC}_{\mathbb{N}}^b \times \mathsf{C}_{\mathbb{N}}$, or, equivalently, that $(\mathsf{IsFinite}^a \times \mathsf{TC}_{\mathbb{N}}^b) * \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{IsFinite}^a \times \mathsf{TC}_{\mathbb{N}}^b \times \mathsf{C}_{\mathbb{N}}$.

Let $\mathsf{minC}_{\mathbb{N}}$ be the single-valued problem whose instances are the instances of $\mathsf{C}_{\mathbb{N}}$, and whose solution for every instance e is the *minimal* n such that $n \notin \mathrm{ran}(e)$. Since it is easy to see that $\mathsf{minC}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$, we can prove that $(\mathsf{IsFinite}^a \times \mathsf{TC}_{\mathbb{N}}^b) * \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{IsFinite}^a \times \mathsf{TC}_{\mathbb{N}}^b \times \mathsf{minC}_{\mathbb{N}}$.

Let $(e, i) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be an instance of $\mathsf{TC}_{\mathbb{N}} * \mathsf{C}_{\mathbb{N}}$, where e is an instance of $\mathsf{C}_{\mathbb{N}}$ and, for every $n \in \mathsf{C}_{\mathbb{N}}(e)$, $\Phi(n, i)$ is an instance of $\mathsf{IsFinite}^a \times \mathsf{TC}_{\mathbb{N}}^b$, where Φ is a universal Turing functional. For notational convenience, we will rephrase this by saying that there are $a+b$ functionals $\Gamma^0, \Gamma^1, \dots, \Gamma^{a-1}$ and $\Delta^0, \Delta^1, \dots, \Delta^{b-1}$ such that, for every $j < a$, every $k < b$ and every $n \in \mathsf{C}_{\mathbb{N}}(e)$, $\Gamma^j(n, i)$ is an instance of $\mathsf{IsFinite}$ and $\Delta^k(n, i)$ is an instance of $\mathsf{TC}_{\mathbb{N}}$.

We now describe the forward functional of the reduction, i.e. we describe how to obtain the instances e'_k of $\mathsf{TC}_{\mathbb{N}}^b$ and p_j of $(\mathsf{IsFinite})^a$ that will be passed to $(\mathsf{IsFinite})^a \times \mathsf{TC}_{\mathbb{N}}^b \times \mathsf{C}_{\mathbb{N}}$.

For every $k < b$, we define the instance e'_k of $\mathsf{TC}_{\mathbb{N}}$ in stages as follows. Let a computable bijection $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given. At step s, we check what (finite) set is being enumerated by $e(s)$, and we take the minimal element of the complement of it, call it n_s. Then, we run $\Delta_s^k(n_s, i)(0), \dots, \Delta_s^k(n_s, i)(s)$, i.e. we run the first s steps of the computation of $\Delta^k(n_s, i)(\ell)$ for every $\ell \leq s$. Let m_s be the minimal number that does not appear in any of the $\Delta_s^k(n_s, i)(\ell)$, and let e' enumerate $\langle s, h \rangle$ for every $h \neq m_s$ (if all of the $\Delta_s^k(n_s, i)(\ell)$ are empty, then let e' enumerate all of the $\langle s, h \rangle$). Moreover, for $s > 0$, if $m_s \neq m_{s-1}$, we let e' enumerate $\langle t, m_{s-1} \rangle$ as well, for all $t < s$. Iterate the procedure for every $s \in \mathbb{N}$.

Similarly, for every $j < a$, we define an instance p_j of $\mathsf{IsFinite}$ in stages as follows: at every stage s, we will define a finite approximation $p_{j,s} \in 2^{<\mathbb{N}}$ of the final string p_j. All of these strings are constituted by two parts, $p_{j,s}^g$ (the "garbage" part of the string) and $p_{j,s}^u$ (the "useful" part of the string), and at every stage $p_{j,s} = p_{j,s}^g {}^\frown p_{j,s}^u$. We start stage 0 by putting $p_{j,0}^g = p_{j,0}^u = \emptyset$. For every stage s, we check what (finite) set is being enumerated by $e(s)$, and we take the minimal element of the complement of it, call it n_s. We run $\Gamma_s^j(n_s, i)(0), \dots, \Gamma_s^j(n_s, i)(s)$ (notice that we can always suppose that $\Gamma^j(n_s, i)(\ell+1) \downarrow \to \Gamma^j(n_s, i)(\ell) \downarrow$). Then, there are two cases.

- If $n_s = n_{s-1}$, we let

$$p_{j,s+1}^g = p_{j,s}^g \text{ and } p_{j,s+1}^u = \Gamma_s^j(n_s, i)(0)^\frown \dots {}^\frown \Gamma_s^j(n_s, i)(s),$$

i.e., we let the "garbage" part of the string unchanged and we extend the "useful" part by the new elements enumerated by Γ^j.

– If instead $n_s \neq n_{s-1}$, we let

$$p^g_{j,s+1} = p^g_{j,s} {}^\frown p^u_{j,s} \text{ and } p^u_{j,s+1} = \Gamma^j_s(n_s,i)(0)^\frown \dots {}^\frown \Gamma^j_s(n_s,i)(s),$$

i.e., we extend the "garbage" part of the string to include what previously was the "useful" part, and we start with a new "useful" part obtained by running the computation $\Gamma^j(n_s,i)$.

We iterate the procedure for every integer s.

After a sufficiently large step s, n_s stabilizes to the actual n such that $n \in \mathsf{minC}_{\mathbb{N}}(e)$, and so $\Gamma^j(n_s,i)$ and $\Delta^k(n_s,i)$ produce actual instances p_j and the e'_k of LPO' and $\mathsf{TC}_{\mathbb{N}}$, respectively (the case of LPO' is maybe less than obvious: the fundamental fact is that, for a sufficiently large stage s, the "garbage" part of every string stops growing, and so we are just extending the "useful" part from that stage on).

Then, for every $j < a$, $\mathsf{IsFinite}(p_j) = \mathsf{IsFinite}(\Gamma^j(n_s,i))$, since p_j is exactly $\Gamma^j(n_s,i))$ plus a finite initial segment (namely, the "garbage" part of the string"). Moreover, for every $k < b$, it is easy to see that, if $\Delta^k(n_s,i)$ enumerates all of $\mathbb{N}$, then so does e', whereas if m is minimal such that $m \notin \operatorname{ran}\left(\Delta^k(n_s,i)\right)$, then e' enumerates all of $\mathbb{N}$ except for numbers of the form $\langle t, m\rangle$, for t large enough.

Hence, if we consider some

$$(c_0,\dots,c_{a-1},d_0,\dots,d_{b-1},n) \in \\ \mathsf{IsFinite}^a \times \mathsf{TC}^b_{\mathbb{N}} \times \mathsf{minC}_{\mathbb{N}}(p_0,\dots,p_{a-1},e'_0,\dots,e'_{b-1},e),$$

we see that $(c_0,\dots,c_{a-1},\pi_2(d_0),\dots,\pi_2(d_{b-1}),n) \in (\mathsf{IsFinite}^a \times \mathsf{TC}^b_{\mathbb{N}}) * \mathsf{C}_{\mathbb{N}}(e,i)$, thus proving the reduction (by π_i we denote the projection on the ith component, so $\langle \pi_1(x), \pi_2(x)\rangle = x$)). □

C The Weihrauch complexity of cShuffle

First, we note that cShuffle is closed under finite products.

Lemma 20. $\mathsf{cShuffle} \times \mathsf{cShuffle} \leq_{\mathrm{W}} \mathsf{cShuffle}$. *Hence,* $\mathsf{cShuffle}^* \leq_{\mathrm{W}} \mathsf{cShuffle}$.

Proof. Let (n_0,f_0) and (n_1,f_1) be instances of cShuffle. Let us fix a computable bijection $\langle\cdot,\cdot\rangle : n_0 \times n_1 \to n_0n_1$ and define the colouring $f\colon \mathbb{Q} \to n_0n_1$ by $f(x) = \langle f_0(x), f_1(x)\rangle$ for every $x \in \mathbb{Q}$. Hence, (n_0n_1, f) is a valid instance of cShuffle. Let $C \in \mathsf{cShuffle}(n_0n_1,f)$: this means that there is an interval I that is a f-shuffle for the colours of C. For $i < 2$, let $C_i := \{j : \exists c \in C(j = \pi_i(j))\}$, where π_i is the projection on the ith component. Then, $C_i \in \mathsf{cShuffle}(n_if_i)$, as witnessed by the interval I. □

We are now ready to gauge the strength of cShuffle. We start with the easy direction.

Lemma 8. $(\mathsf{LPO}')^* \leq_{\mathrm{W}} \mathsf{cShuffle}$

Proof. Thanks to Lemma 20, it is enough to prove that $\mathsf{LPO}' \leq_{\mathrm{W}} \mathsf{cShuffle}$.

We will prove that $\mathsf{IsFinite} \leq_{\mathrm{sW}} \mathsf{cShuffle}$. Let $p \in 2^{\mathbb{N}}$ be an infinite binary string, we define a colouring of the rationals as follows: we define a colouring of the dyadic numbers $c\colon \mathbb{D} \to 2$, and, using the fact that there exists a computable order-preserving bijection $q : \mathbb{Q} \to \mathbb{D}$, we will then consider $c \circ q$, and apply $\mathsf{cShuffle}$ to that colouring.

We now construct c by setting $c(d) = p(n)$ for every $d \in \mathbb{D}$ with $\mathrm{rank}(d) = n$. Apply $\mathsf{cShuffle}$ to $c \circ q$, and let $C \in \mathsf{cShuffle}(c \circ q)$. Since density implies infinity, if $1 \in C$, then p had infinitely many occurrences of 1. On the other hand, if for infinitely many n $p(n) = 1$, then all the dyadics of the form $\frac{a}{2^n}$ are coloured 1 by c, which implies that the colour 1 occurs densely in every interval. Hence, $1 \in C$ if and only if 1 appeared in p infinitely often, which proves the claim. □

Next, we move to the more difficult reduction.

Lemma 9. *Let* $\mathsf{cShuffle}_n$ *be the restriction of* $\mathsf{cShuffle}$ *to the instances of the form* (n, c)*. Then,* $\mathsf{cShuffle}_n \leq_{\mathrm{W}} (\mathsf{LPO}')^{2^n-1}$

Proof. We actually show that $\mathsf{cShuffle}_n \leq_{\mathrm{W}} \mathsf{IsFinite}^{2^n-1}$. Let (n, c) be an instance of $\mathsf{cShuffle}$. The idea is that we will use one instance of $\mathsf{IsFinite}$ for every non-empty subset C of the set of colours n, in order to determine for which such Cs there exists an interval I_C such that $c(I_C) = C$. We will then prove that any $\subseteq$-minimal such C is a solution for (n, c).

Let C_i, for $i < 2^n - 1$, be an enumeration of the non-empty subsets of n. Let I_j be an enumeration of the open intervals of $\mathbb{Q}$, and let q_h be an enumeration of $\mathbb{Q}$. For every $i < 2^n - 1$, we build an instance p_i of $\mathsf{IsFinite}$ in stages in parallel. At every stage s, for every component $i < 2^n - 1$, there will be a "current interval" $I_{j^i_s}$ and a "current point" $q_{h^i_s}$. We start the construction by setting the current interval to I_0 and the current point to q_0 for every component i.

For every component i, at stage s we do the following:

- if $q_{h^i_s} \notin I_{j^i_s}$ or if $c(q_{h^i_s}) \in C_i$, we set $I_{j^i_{s+1}} = I_{j^i_s}$ and $q_{h^i_{s+1}} = q_{h_s+1}$. Moreover, we set $p_i(s) = 0$. In practice, this means that if the colour of the current point is in C_i, or if the current point is not in the current interval, no special action is required, and we can move to consider the next point.
- If instead $q_{h^i_s} \in I_{j^i_s}$ and $c(q_{h^i_s}) \notin C_i$, we set $I_{j^i_{s+1}} = I_{j_s+1}$ and $q_{h^i_{s+1}} = q_0$. Moreover, we set $p_i(s) = 1$. In practice, this means that if the current point is in the current interval and its colour is not a colour of C_i, then, we need to move to consider the next interval in the list, and therefore we reset the current point to the first point in the enumeration. Moreover, we record this event by letting $p_i(s)$ take value 1.

We iterate the construction for every $s \in \mathbb{N}$. After infinitely many steps, we obtain an instance $\langle p_0, p_1, \ldots, p_{2^n-2} \rangle$ of $\mathsf{IsFinite}^{2^n-1}$. Let $\sigma \in 2^{2^n-1}$ be such that $\sigma \in \mathsf{IsFinite}^{2^n-1}(\langle p_0, p_1, \ldots, p_{2^n-2} \rangle)$.

To find a set of colours C for which there is a c-shuffle, we proceed s follows. We start checking $\sigma(i)$ for i such that C_i is a singleton: if, for any such i, $\sigma(i) = 1$, it means that the corresponding p_i has only finitely many 1s, which implies that the second case in the construction was triggered only finitely many times. Hence, there is a stage s such that, for every $t > s$, $I_{j_s^i} = I_{j_t^i}$. This means that $I_{j_s^i}$ is c-homogeneous, and thus, in particular, a c-shuffle. Hence, C_i is a valid solution.

If instead for all is such that C_i is a singleton $\sigma(i) = 0$: then, we know that no interval I is c-monochromatic, otherwise we would be in the previous case. We move to consider the is such that $|C_i| = 2$. Suppose that for one such i, $\sigma(i) = 1$: again, this means that, for a sufficiently large stage s, the current interval $I_{j_s^i}$ is such that, for every $q \in I_{j_s^i}$, $c(q) \in C_i$, since the second case in the construction is triggered only finitely many times. But since we know that there are no c-monochromatic intervals, the two colours of C_i occur densely in $I_{j_s^i}$, which then is a c-shuffle for the colours in C_i. Hence, any C_i such that $\sigma(i) = 1$ is a valid solution for c.

This argument can be iterated for every number of colours. Since, by the theory, a c-shuffle exists, at least one of the p_i instances above contains only finitely many 1s. To compute a solution to c, it is thus sufficient to look for the minimal k such that, for some i, $\sigma(i) = 1$ and $|C_i| = k$, and output C_i. □

Putting the two previous results together, we have the following.

Theorem 4. $(\mathsf{LPO}')^* \equiv_{\mathrm{W}} \mathsf{cShuffle}$

Proof. $(\mathsf{LPO}')^* \leq_{\mathrm{W}} \mathsf{cShuffle}$ is given directly by Lemma 8. For the other direction, notice that $\mathsf{cShuffle} \equiv_{\mathrm{W}} \prod_{n \in \mathbb{N}} \mathsf{cShuffle}_n$. The result then follows from Lemma 9. □

D Proof of Lemma 10

Lemma 10. *Let* $\mathsf{iShuffle}_n$ *be the restriction of* $\mathsf{iShuffle}$ *to the instances of the form* (n, c)*. For every* $n \in \mathbb{N}$ *with* $n \geq 2$*,* $\mathsf{iShuffle}_n \leq_{\mathrm{sW}} \mathsf{TC}_{\mathbb{N}}^{n-1}$.

Proof. Fix an enumeration I_j of the intervals of $\mathbb{Q}$, an enumeration q_h of $\mathbb{Q}$, a computable bijection $\langle \cdot, \cdot \rangle \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and let (n, c) be an instance of $\mathsf{iShuffle}_n$.

The idea of the reduction is the following: with the first instance e_{n-1} of $\mathsf{TC}_{\mathbb{N}}$, we look for an interval I_j on which c takes only $n-1$ colours: if no such interval exists, then this means that every colour is dense in every interval, and so every I_j is a valid solution to c. Hence, we can suppose that such an interval is eventually found: we will then use the second instance e_{n-2} of $\mathsf{TC}_{\mathbb{N}}$ to look for a subinterval of I_j where c takes only $n-2$ values. Again, we can suppose that such an interval is found. We proceed like this for $n-1$ steps, so that in the end the last instance e_1 of $\mathsf{TC}_{\mathbb{N}}$ is used to find an interval I' inside an interval I on which we know that at most two colours appear: again, we look for c-monochromatic intervals: if we do not find any, then I' is already a c-shuffle, whereas if we do find one, then that interval is now a solution to c, since c-monochromatic intervals are trivially c-shuffles..

More formally, we proceed as follows: we define $n-1$ instances $e_1, \dots, e_{n-1}$ of $\mathsf{TC}_{\mathbb{N}}$ as follows. For every stage s, every instance e_i will have a "current interval" $I_{j^i_s}$ and a "current point" $q_{h^i_s}$ and a "current list of colours" $L_{k^i_s}$. We start the construction by the setting the current interval equal to I_0, the current point equal to q_0 and the current list of points equal to $\emptyset$ for every i.

At stage s, there are two cases:

- if, for every i, $q_{h^i_s} \notin I_{j^i_s}$ or $|L_{k^i_s} \cup \{c(q_{h^i_s})\}| \leq i$, we set $I_{j^i_{s+1}} = I_{j^i_s}$, $q_{h^i_{s+1}} = q_{h^i_s+1}$ and $L_{k^i_{s+1}} = L_{k^i_s} \cup \{c(q_{h^i_s})\}$. Moreover, we let every e_i enumerate every number of the form $\langle s, a\rangle$, for every $a \in \mathbb{N}$, except for $\langle s, j^i_s\rangle$. We then move to stage $s+1$.
  In practice, this means that if the set of colours of the points of the current interval seen so far does not have cardinality larger than i, no particular action is required, and we can move to check the next point on the list.
- otherwise: let i' be maximal such that $q_{h^i_s} \in I_{j^i_s}$ and $|L_{k^i_s} \cup \{c(q_{h^i_s})\}| > i$. Then, for every $i > i'$ we proceed as in the previous case (i.e., the current interval, current point, current list of colours and enumeration are defined as above). For the other components, we proceed as follows: we first look for the minimal $\ell > j^{i'}_s$ such that $I_\ell \subseteq I_{j^{i'}_s+1}$ (if $i' = n-1$, just pick $\ell = j^{n-1}_s + 1$). Then, for every $i \leq i'$, we set $I_{j^i_{s+1}} = I_\ell$, $q_{h^i_{s+1}} = q_0$ and $L_{k^i_{s+1}} = \emptyset$. Moreover, we let e_i enumerate every number of the form $\langle t, a\rangle$ with $t < s$ that had not been enumerated so far, and also every number of the form $\langle s, a\rangle$, with the exception of $\langle s, j^i_s\rangle$. We then move to stage $s+1$.
  In practice, this means that if, for a certain component i', we found that the current interval has too many colours, then, for all the components $i \leq i'$, we move to consider intervals strictly contained in the current interval of component i'.

We iterate the procedure for every $s \in \mathbb{N}$, thus obtaining the $\mathsf{TC}^{n-1}_{\mathbb{N}}$-instance $\langle e_1, \dots, e_{n-1}\rangle$.

Let $\sigma \in \mathbb{N}^{n-1}$ be such that $\sigma \in \mathsf{TC}^{n-1}_{\mathbb{N}}(\langle e_1, \dots, e_{n-1}\rangle)$. Then, we look for the minimal i such that $I_{\pi_2(\sigma(i))} \subseteq I_{\pi_2(\sigma(i+1))} \subseteq \dots \subseteq I_{\pi_2(\sigma(n-1))}$ (by π_i we denote the projection on the ith component, so $\langle \pi_1(x), \pi_2(x)\rangle = x$)). We claim that $I_{\pi_2(\sigma(i))}$ is a c-shuffle, which is sufficient to conclude that $\mathsf{iShuffle}_n \leq_{\mathrm{sW}} \mathsf{TC}^{n-1}_{\mathbb{N}}$.

We now prove the claim. First, suppose that e_{n-1} enumerates all of $\mathbb{N}$. Then, the second case of the construction was triggered infinitely many times with $i' = n-1$: hence, no interval contains just $n-1$ colours, and so, as we said at the start of the proof, this means that every interval is a c-shuffle. In particular, this imples that $I_{\pi_2(\sigma(i))}$ is a valid solution. Hence we can suppose that e_{n-1} does not enumerate all of $\mathbb{N}$.

Next, we notice that for every $m > 1$, if e_m enumerates all of $\mathbb{N}$, the so does e_{m-1}, by inspecting the second case of the construction. Let m be minimal such that e_m does not enumerate all of $\mathbb{N}$. For such an m, it is easy to see that $I_{\pi_2(\sigma(m))}$ is a valid solution to c: indeed, we know from the construction that c takes m colours on $I_{pi_2(\sigma(m))}$, and that for no interval contained in $I_{\pi_2(\sigma(m))}$ c takes $m-1$ colours, which means that $I_{\pi_2(\sigma(m))}$ is a c-shuffle. Moreover, it is easy to see that $I_{\pi_2(\sigma(m))} \subseteq I_{\pi_2(\sigma(m+1))} \subseteq \dots I_{\pi_2(\sigma(n-1))}$, which implies that $i \leq m$. Since every subinterval of a c-shuffle is a c-shuffle, $I_{\pi_2(\sigma(i))}$ is a valid solution to c, as we wanted. □

E Proof of Lemma 13

Lemma 13. $\mathsf{Shuffle} \times \mathsf{Shuffle} \leq_{\mathrm{W}} \mathsf{Shuffle}$. *Therefore,* $\mathsf{Shuffle}^* \equiv_{\mathrm{W}} \mathsf{Shuffle}$.

Proof idea. The Lemma is proved exactly as Lemma 20. □

F Proof of Lemma 15

Lemma 15. *Let* $\mathsf{Shuffle}_n$ *be the restriction of* $\mathsf{Shuffle}$ *to the instances of the form* (n, c)*. Then,* $\mathsf{Shuffle}_n \leq_{\mathrm{W}} (\mathsf{TC}_{\mathbb{N}} \times \mathsf{LPO}')^{2^n-1}$

Proof. Let (n, c) be an instance of $\mathsf{Shuffle}$. The idea of the proof of $\mathsf{Shuffle}_n \leq_{\mathrm{W}} (\mathsf{TC}_{\mathbb{N}} \times \mathsf{LPO}')^{2^n-1}$ is, in essence, to combine the proofs of Lemma 10 and of Theorem 4: we want to use $\mathsf{TC}_{\mathbb{N}}$ to find a candidate interval for a certain subset C of n, and on the side we use LPO' (or equivalently, $\mathsf{IsFinite}$) to check for every such set C whether a c-shuffle for the colours of C actually exists. The main difficulty with the idea described above is that the two proofs must be intertwined, in order to be able to find both a c-shuffle and the set of colours that appears on it.

We proceed as follows: let C_i be an enumeration of the non-empty subsets of n. Moreover, let us fix some computable enumeration I_j of the intervals of $\mathbb{Q}$, some computable enumeration q_h of the points of $\mathbb{Q}$, and some computable bijection $\langle \cdot, \cdot \rangle \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. For every C_i, we will define an instance $\langle p_i, e_i \rangle$ of $\mathsf{IsFinite} \times \mathsf{TC}_{\mathbb{N}}$ in stages as follows: at every stage s, for every index i, there will be a "current interval" $I_{j^i_s}$ and a "current point" $q_{h^i_s}$. We begin stage 0 by setting the current interval to I_0 and the current point to q_0 for every index i.

At stage s, for every component i, there are two cases:

- if $q_{h^i_s} \notin I_{j^i_s}$ or if $c(q_{h^i_s}) \in C_i$, we set $I_{j^i_{s+1}} = I_{j^i_s}$ and $q_{h^i_{s+1}} = q_{h^i_s+1}$. Moreover, we set $p_i(s) = 0$ and we let e_i enumerate all the integers of the form $\langle s, a \rangle$, except $\langle s, j^i_{s+1} \rangle$. We then move to stage $s + 1$.
  In plain words, for every component i, we check if the colour of the current point is in C_i, or if the current point is not in the current interval: if this happens, no special action is required.
- If instead $q_{h^i_s} \in I_{j^i_s}$ and $c(q_{h^i_s}) \notin C_i$, we set $I_{j^i_{s+1}} = I_{j^i_s+1}$ and $q_{h^i_{s+1}} = q_0$. Moreover, we set $p_i(s) = 1$, and we let e_i enumerate all the numbers of the form $\langle t, a \rangle$, for $t < s$, that had not been enumerated at a previous stage, and also all the numbers of the form $\langle s, a \rangle$, with the exception of $\langle s, j^i_{s+1} \rangle$. We then move to stage $s + 1$.
  In plain words: if we find that for some component i the colour of the current point is not in C_i, then, from the next stage, we start considering another

interval, namely the next one in the fixed enumeration. We then reset the current point to q_0 (so that all rationals are checked again), and we record the event by letting $p_i(s) = 1$ and changing the form of the points that e_i is enumerating.

We iterate the procedure for every integer s. Let $\sigma \in (2 \times \mathbb{N})^{2^n-1}$ be such that

$$\sigma \in (\mathsf{IsFinite} \times \mathsf{TC}_{\mathbb{N}})^{2^n-1}(\langle\langle p_1, e_1\rangle \dots, \langle p_{2^n-1}, e_{2^n-1}\rangle\rangle)$$

Let k be the minimal cardinality of a subset $C_i \subseteq n$ such that $\mathsf{IsFinite}(p_i) = 1$: notice that such a k must exist, because c-shuffle exist. Then, we claim that the corresponding $I_{\pi_2(\sigma(i))}$ is a c-shuffle (by π_i we denote the projection on the ith component, so $\langle \pi_1(x), \pi_2(x)\rangle = x$)). If we do this, it immediately follows that $\mathsf{Shuffle} \le_{\mathrm{W}} ((\mathsf{LPO}') \times \mathsf{TC}_{\mathbb{N}})^{2^n-1}$.

Hence, all that is left to be done is to prove the claim. By the fact that $\mathsf{IsFinite}(p_i) = 1$, we know that the second case of the construction is triggered only finitely many times. Hence, e_i does not enumerate all of $\mathbb{N}$, and so $I_{\pi_2(\sigma(i))}$ is an interval containing only colours from C_i. Moreover, by the minimality of $|C_i|$, we know that no subinterval of $I_{\pi_2(\sigma(i))}$ contains fewer colours, which proves that $I_{\pi_2(\sigma(i))}$ is a c-shuffle. ☐

G Proof of Theorem 8

Theorem 8. *Let $\mathsf{ORT}_{\mathbb{Q}}$ be the problem whose instances are ordered colourings $c : [\mathbb{Q}]^2 \to P$, for some finite poset $(P, \prec)$, and whose possible outputs on input c are intervals on which c is constant. We have that $\mathsf{ORT}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{LPO}^*$.*

Proof. $\mathsf{LPO}^* \le_{\mathrm{sW}} \mathsf{ORT}_{\mathbb{Q}}$: let $\langle n, p_0, \dots, p_{n-1}\rangle$ be an instance of LPO^*. Let $(P, \prec)$ be the poset such that $P = 2^n$, the set of subsets of n, and $\prec = \supset$, i.e. $\prec$ is reverse inclusion.

We define an ordered colouring $c : [\mathbb{Q}]^2 \to P$ in stages by deciding, at stage s, the colour of all the pairs of points $(x, y) \in [\mathbb{Q}]^2$ such that $|x - y| > 2^{-s}$.

At stage 0, we set $c(x, y) = \emptyset$ for every $(x, y) \in [\mathbb{Q}]^2$ such that $|x - y| > 1$. At stage $s > 0$, we check $p_i\big|_s$ for every $i < n$ (i.e., for every i, we check the sequence p_i up to $p_i(s - 1)$), and for every $(x, y) \in [\mathbb{Q}]^2$ with $2^{-s+1} \ge |x - y| > 2^{-s}$, we let

$$c(x, y) = \{i < n : \exists t < s(p_i(t) = 1)\}.$$

It is easily seen that c defined as above is an ordered colouring: if $x \le x' < y' \le y'$, then $|x' - y'| \le |x - y|$, which means that to determine the colour of (x', y') we need to examine a longer initial segment of the p_is. Let $I \in \mathsf{ORT}_{\mathbb{Q}}(P, c)$, and let $\ell \in \mathbb{N}$ be least such that the length of I is larger that $2^{-\ell}$: since I is c-homogeneous, we know that for every $i < n$, $\exists t(p_i(t) = 1) \Leftrightarrow \exists t < \ell(p_i(t) = 1)$. Hence, for every pair of points $(x, y) \in [I]^2$, the colour of $c(x, y)$ is exactly the set of i such that $\mathsf{LPO}(p_i) = 1$.

$\mathsf{ORT}_{\mathbb{Q}} \le_{\mathrm{W}} \mathsf{LPO}^*$: Let (P, c) be an instance of $\mathsf{ORT}_{\mathbb{Q}}$, for some finite poset $(P, \prec_P)$. Let $<_L$ be a linear extension of $\prec_P$, and notice that $c : \mathbb{Q} \to (P, <_L)$ is still an ordered colouring. Let $r_0 <_L r_1 <_L \cdots <_L r_{|P|-1}$ be the elements of P. The idea of the proof is to have one instance of LPO per element of P, and to check in parallel the intervals of the rationals to see if they are c-homogeneous for the corresponding element of P. Anyway, one has to be careful as to how these intervals are chosen: to give an exampe, if we find that a certain interval I is not c-homogeneous for the $<_L$-maximal element $r_{|P|-1}$, because we found, say, $x < y$ such that $c(x, y) \neq r_{|P|-1}$, then not only do we flag the corresponding instance of LPO by letting it contain a 1, but we also restrict the research of all the other components so that they only look at intervals *contained in* $]x, y[$. By proceeding similarly for all the components, since c is ordered, we are sure that we will eventually find a c-homogeneous interval.

We define the $|P|$ instances $p_0, p_1, \ldots, p_{|P|-1}$ of LPO in stages as follows. Let a_n be an enumeration of the ordered pairs of rationals, i.e. an enumeration of $[\mathbb{Q}]^2$, with infinitely many repetitions. At every stage s, some components i will be "active", whereas the remaining components will be "inactive": if a component i is inactive, it can never again become active. Moreover, at every stage s, there is a "current pair" a_{n_s} and a "current interval" a_{m_s} (for this proof, it is practical to see ordered pairs of rational as both pairs and as denoting extrema of an open interval). We begin stage 0, by putting the current pair and the current interval equal to a_0. Moreover, every component is set to be active.

At stage s, for every inactive component $j < |P|$, we set $p_j(s) = 1$. For every active component i, there are two cases:

- if, for every active component i, $c(a_{n_s}) \geq_L r_i$, then we look for the smallest $\ell > n_s$ such that $a_\ell \subseteq a_{m_s}$ (i.e., we look for a pair of points contained in the current interval), and set $a_{n_{s+1}} = a_\ell$, and $a_{m_{s+1}} = a_{m_s}$. We set $p_i(s) = 0$ and no component is set to inactive. We then move to stage $s + 1$.
- suppose instead there is an active component i such that $c(a_{n_s}) <_L r_i$: let i be the minimal such i, then we set every $j \geq i$ to inactive (the ones that were already inactive remain so) and we let $p_j(s) = 1$. We then let $a_{m_{s+1}} = a_{n_s}$, and we look for the least $\ell > n_s$ such that $a_\ell \subset a_{n_s}$: we set $a_{n_{s+1}} = a_\ell$, and we set $p_k(s) = 0$ for every active component $k < |P|$. We then move to stage $s + 1$.

We iterate the procedure above for every integer s.

Let $\sigma \in 2^{|P|}$ be such that $\sigma \in \mathsf{LPO}^*(\langle |P|, p_0, \ldots, p_{|P|-1} \rangle)$. Notice that $\sigma(0) = 0$, since no pair of points can attain colour $<_L$-below r_0. Moreover, notice that $\sigma(i) = 0$ if and only if the component i was never set inactive. Hence, let i be maximal such that $\sigma(i) = 0$, and let t be a state such all components $j > i$ have been set inactive by step t. Hence, after step t, the current interval I never changes, and thus we eventually check the colour of all the pairs in that interval. Since the second case of the construction is never triggered, it follows that I is a c-homogeneous interval. Hence, in order to find it, we know we just have to repeat the construction above until all the components of index larger than i are set inactive. This proves that $\mathsf{ORT}_{\mathbb{Q}} \leq_{\mathrm{W}} \mathsf{LPO}^*$. □

H Proof of Theorem 9

Theorem 9.
- $\mathsf{cART}_{\mathbb{Q}} \leq_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{LPO}^*$, *therefore* $\mathsf{cART}_{\mathbb{Q}} \equiv_{\mathrm{W}} (\mathsf{LPO}')^*$.
- $\mathsf{iART}_{\mathbb{Q}} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^* \times \mathsf{LPO}^*$, *therefore* $\mathsf{iART}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$.
- $\mathsf{ART}_{\mathbb{Q}} \leq_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^* \times \mathsf{LPO}^*$, *therefore* $\mathsf{ART}_{\mathbb{Q}} \equiv_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^*$.

Proof. The three results are all proved in a similar manner. Before starting the proof, we recall that $\mathsf{LPO}^* \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$, which is a known fact. This enables us to use Lemma 4 with LPO^* in place of P.

For $\mathsf{x} \in \{\mathsf{c}, \mathsf{i}, \mathsf{s}\}$ and every $n \in \mathbb{N}$, let $\mathsf{xART}_{\mathbb{Q},n}$ be the restriction of $\mathsf{xART}_{\mathbb{Q}}$ to instances of the form (S, c) with S of cardinality n. Hence, by the considerations preceding the statement of the theorem in the body of the paper, we have the following facts:

- $\mathsf{cART}_{\mathbb{Q},n} \leq_{\mathrm{W}} \mathsf{cShuffle}_{n^2} * \mathsf{ORT}_{\mathbb{Q}}$, hence, by Lemma 9 and Theorem 8, we have that $\mathsf{cART}_{\mathbb{Q},n} \leq_{\mathrm{W}} (\mathsf{LPO}')^{2^{n^2}-1} * \mathsf{LPO}^*$. By Lemma 4, we have that $\mathsf{cART}_{\mathbb{Q},n} \leq_{\mathrm{W}} (\mathsf{LPO}')^{2^{n^2}-1} \times \mathsf{LPO}^*$, from which the claim follows.
- $\mathsf{iART}_{\mathbb{Q},n} \leq_{\mathrm{W}} \mathsf{iShuffle}_{n^2} * \mathsf{ORT}_{\mathbb{Q}}$, hence, by Lemma 10 and Theorem 8, we have that $\mathsf{iART}_{\mathbb{Q},n} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{n^2-1} * \mathsf{LPO}^*$. By Lemma 4, we have that $\mathsf{iART}_{\mathbb{Q},n} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{n^2-1} \times \mathsf{LPO}^*$, from which the claim follows.
- $\mathsf{ART}_{\mathbb{Q},n} \leq_{\mathrm{W}} \mathsf{Shuffle}_{n^2} * \mathsf{ORT}_{\mathbb{Q}}$, hence, by Lemma 15 and Lemma 4, we have that $\mathsf{ART}_{\mathbb{Q},n} \leq_{\mathrm{W}} (\mathsf{LPO}' \times \mathsf{TC}_{\mathbb{N}})^{2^{n^2}-1} * \mathsf{LPO}^*$. By Lemma 4, we have that $\mathsf{ART}_{\mathbb{Q},n} \leq_{\mathrm{W}} (\mathsf{LPO}' \times \mathsf{TC}_{\mathbb{N}})^{2^{n^2}-1} \times \mathsf{LPO}^*$, from which the claim follows.

□