CSCM12: software concepts and efficiency
Trees & friends

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March 13th 2023

Logistics

- Friday and next Monday: unless I announce otherwise, **I am on strike** and **Tonicha won't be there either**
- Like last time, I will merely point at a list of exercises in the books that you might want to look at
- Feel free to come say hello on the picket line
- But otherwise, also feel free to progress on the labs and coursework.

Today

Introduce tree-like datastructure

- What are they?
- How to encode them in java?
- Motivating examples & applications

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- What are they?
- How to encode them in java?
- Motivating examples & applications

Also an opportunity to **recap material on sorting algorithms** with heapsort.

(c.f. challenge task of last lab)

Motivations

What do you mean by a tree-like datastructure?

Recursive definition of a list

A list of As is either

- An empty list
- Or an A and (a reference to) another list of As

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Recursive definition of a list

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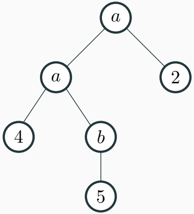
For the most part

Recursive datatypes with possibly **multiple** subobjects of the same kind



versus

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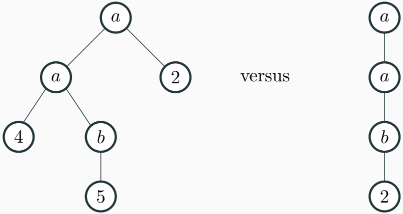
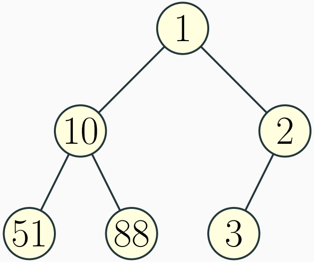


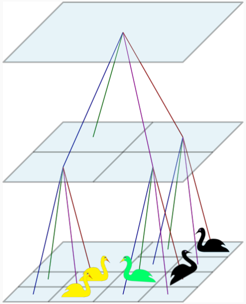
(list-like datatypes are degenerate tree-like datatypes)

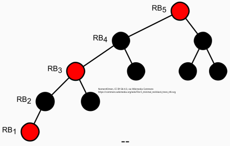
Why should we care? (1/2)

Tree-like structures come up in a variety of contexts:

- Efficient datastructures: sets/priority queues with $\mathcal{O}(\log(n))$ operations, random access lists, quad/octtrees...





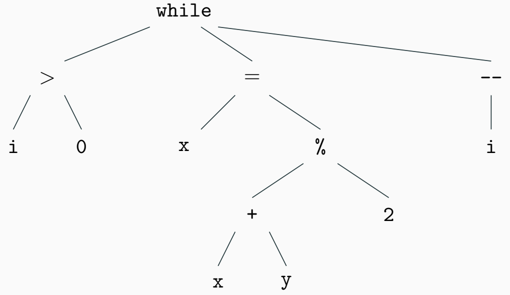


Why should we care? (2/2)

Trees can also come up as natual objects we'd like to manipulate

- E.g., anything hierarchical, abstract syntax trees, directory trees

```
while(i > 0) {
  x = x + y % 2;
  i--;
}
```



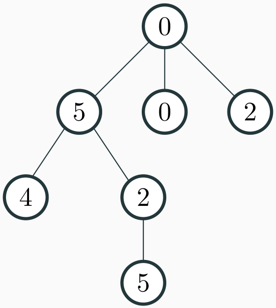
Generalities

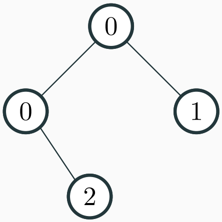
Recursive mathsy definition of a tree

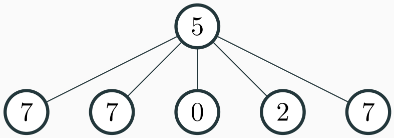
Formal definition

A tree with labels in L is a pair $(\text{label}, \langle c_1, \ldots, c_n \rangle)$ where:

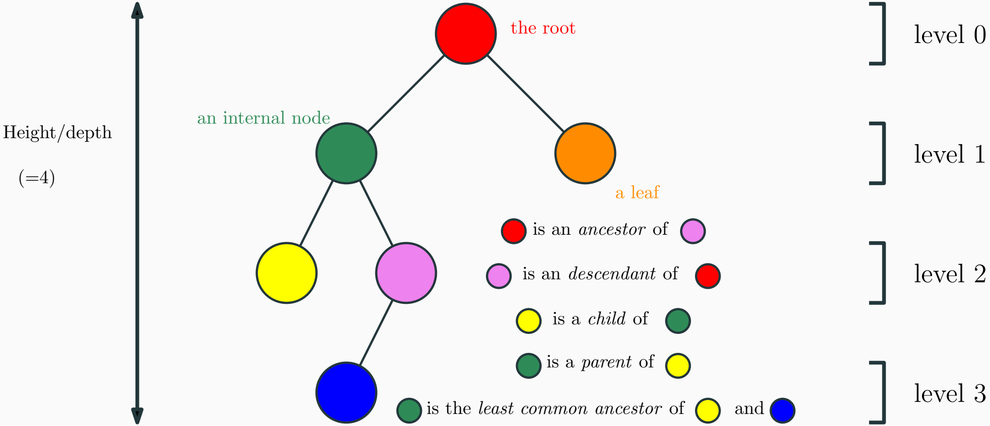
- $\text{label} \in L$
- $\langle c_1, \ldots, c_n \rangle$ is a list of trees with labels in L (possibly an empty list)





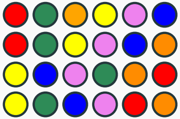


Vocabulary/basic notions



depth $\leq$ size $\leq \max(arity)^{\text{depth}}$

- breadth-first enumeration:
- depth-first prefix enumeration:
- depth-first postfix enumeration:
- depth-first infix enumeration:



(← only makes sense for *binary* trees)

In java

- Typically encoded via a recursive class

```java
class Tree<T>
{
  public T label;
  public ArrayList<Tree<T>> children;

  public static <T> Tree<T> Leaf(T x)
  {
      Tree<T> t = new Tree<T>();
      t.children = new ArrayList<Tree<T>>();
      t.label = x;
      return t;
  }
}
```

```java
Tree<Integer> root = Leaf(10);
root.children.add(Leaf(5));
root.children.add(Leaf(11));
root.children.add(Leaf(12));
root.children.get(1).add(Leaf(3));
```

What tree is this?

Tree example with memory representation

```
Tree<Integer> root = Leaf(10);
root.children.add(Leaf(5));
root.children.add(Leaf(11));
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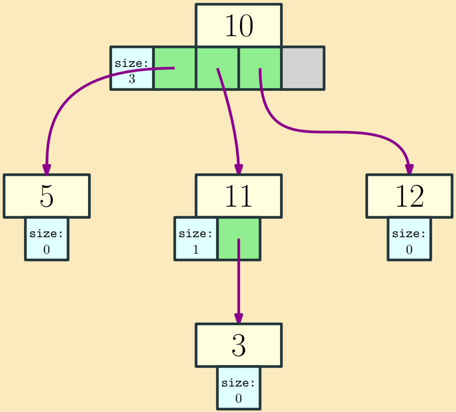
A somewhat honest of the representation in memory

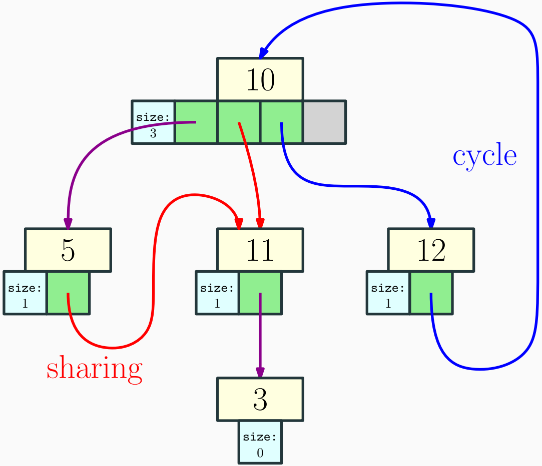


Non-trees?

- With linked lists, possible to create **cycles**

  (and worse pathologies in the case of doubly-linked lists)

- Same here + an additional pitfall: **sharing**



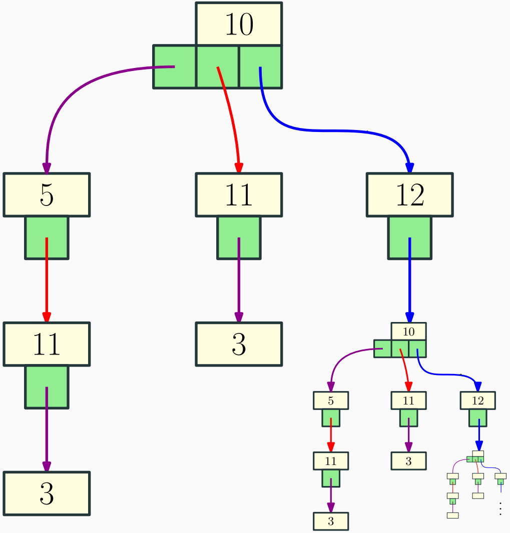
Are those illegitimate?

Not necessarily:

- They can be seen as **graphs**

  (Topic of next lecture)

- Can represent (potentially infinite) trees



A time and place for sharing and cycles

Cons

- **Cycles**: no longer a finite well-defined notion of **depth**

  $\Rightarrow$ a lot of tree algorithm no longer terminates like e.g. traversal

- **Sharing**: a single update modifies **several spots** in the unravelling

  ($\Rightarrow$ not an issue for **immutable** datastructures)

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- **Cycles**: can represent infinite trees in finite space

  only regular trees, which may arguably admit more convenient representations

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A time and place for sharing and cycles

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- One just has to be extra clear about what they consider legal inputs/outputs

A time and place for sharing and cycles

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- Commonplace tacit assumption: No sharing/cycles in tree-like datastructure
- One just has to be extra clear about what they consider legal inputs/outputs
- **For this lecture:** no more sharing/cycles

Tree-like datastructures

Often, you may want more/less flexibility than the generic tree datastructure

- Do you want to bound the arity of internal nodes?
- Do you care about the ordering of children?
- Do you care about empty spots for future children?
- Do you want more labels?
- Do you want different type of labels for e.g. leaves?
- ...

```
class AST {
  boolean isAnOperand;
  String repr;
  AST lhs;
  AST rhs;
}
```

→ for most situations, similar issues/resolutions

One last restriction for today

Binary trees are those trees whose nodes have at most two children.

```
class BTree<T> {
  T label;
  BTree<T> leftChild;
  BTree<T> rightChild;
}
```

Conventions:

- `leftChild` and `rightChild` may be set to `null`

  (for a leaf: both are `null`)

- it is possible that `leftChild = null` and `rightChild != null`

  (we care about the order and “empty spots”)

Binary search trees

Motivation: set with $\mathcal{O}(\log(n))$ lookup and delete

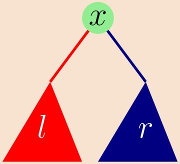
```
Set(); // creates an empty set
void remove(T e); // removes one element
boolean contains(T e); // do I contain the element?
void add(T e); // add one element
Set union(Set s2); // adds all elements of s2
...
```

Op \Data	`Array`	`List`	`ArrayList`	`TreeSet`
`Set(T)`	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
`remove`	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(\log(n))$
`contains`	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(\log(n))$
`add`	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(n)$ $\mathcal{O}(1)$ amortized	$\mathcal{O}(\log(n))$
`union`	$\mathcal{O}(n+m)$	$\mathcal{O}(1)$	$\mathcal{O}(n+m)$ $\mathcal{O}(m)$ amortized	$\mathcal{O}(m\log(n))$

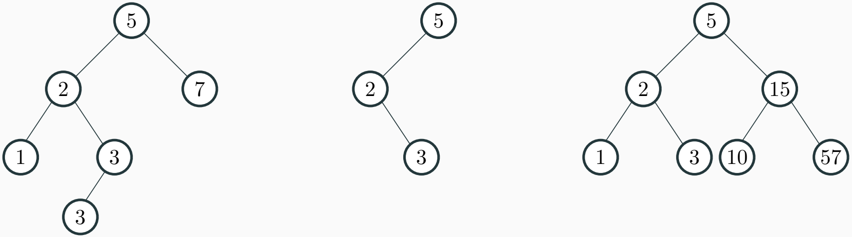
A datastructure to represent set of numbers

Definition

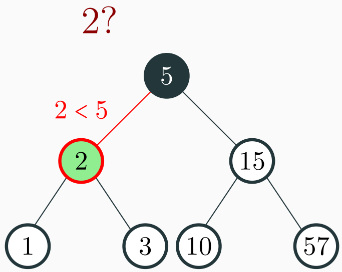
A **Binary Search Tree** is a binary tree labeled by integers such that

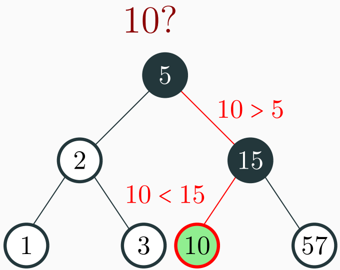


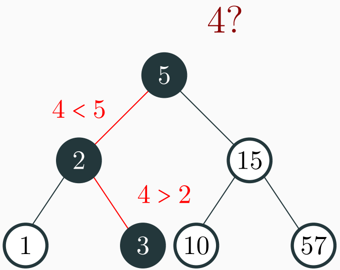
$$\Longrightarrow \qquad l \leq x \leq r$$



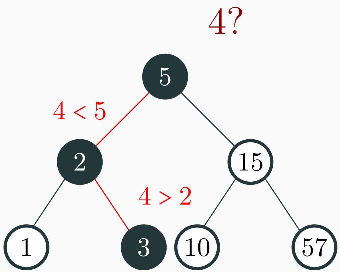
Look up an element `boolean contains(int e)`







Complexity of `boolean contains(int e)`?



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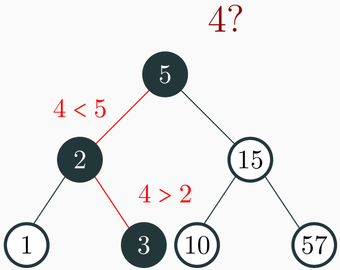

$\rightarrow \mathcal{O}(\text{depth})$

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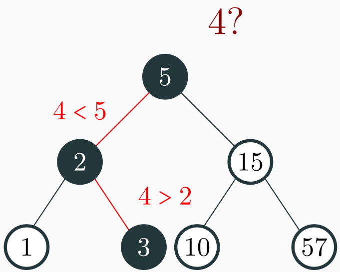


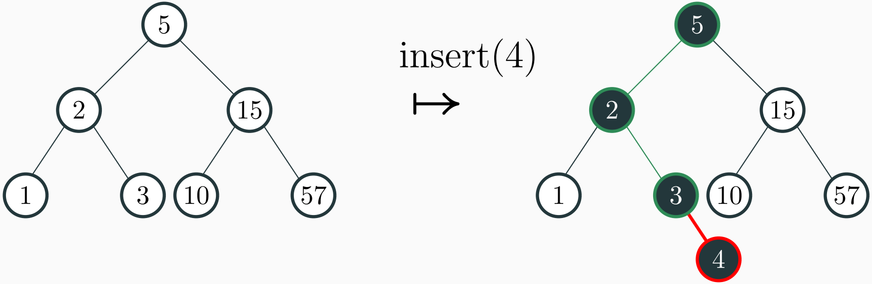
$\rightarrow \mathcal{O}(\text{depth})$

Relation to the size of the set

- Best case: the tree is balanced $\rightarrow$ depth $= \mathcal{O}(\text{size})$
- Worst case: one child everywhere $\rightarrow$ depth $= \Omega(\text{size})$

$\rightarrow$ **Important concern:** work on **balanced trees**

Insert an element `void add(int e)`



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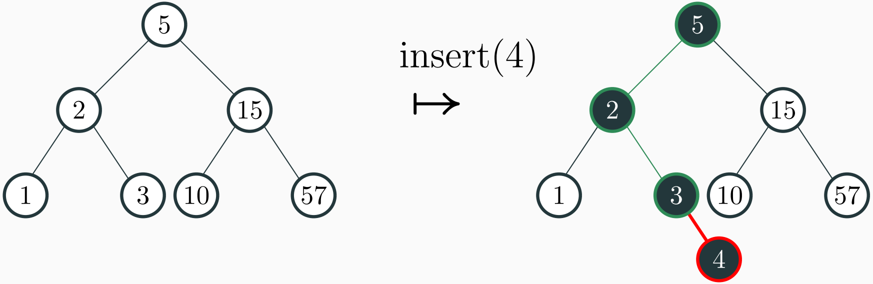
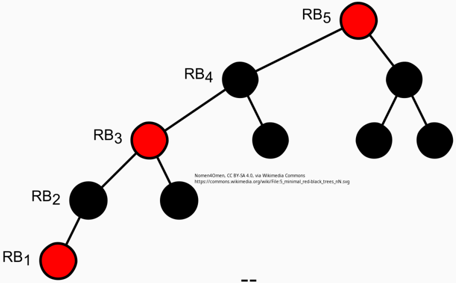


- Complexity: still $\mathcal{O}(\text{depth})$
- **Issue:** repeatedly inserting bigger and bigger elements can unbalance a tree

Try inserting $1, 2, 3, \ldots$ to `Leaf(0)`

Solutions (not covered in-depth here)

- Either try to do some probabilistic analysis and try to prove things are not that bad on average for a given use-case ...
- ... or use fancier invariants to have classes of trees with depth $= \mathcal{O}(\log(\text{size}))$
- Paradigmatic examples: red-black trees and AVLs



- Involved "repair" procedures to maintain the invariants after an insertion/deletion running in $\mathcal{O}(\text{depth})$
- Something like this is implemented for `TreeSet`

So now, you should be able to tell why this table is like this

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Priority queues, heaps and heapsort

Quick note

Motivation

Implement a priority queue with $\mathcal{O}(\log(n))$ operations
$+ \rightarrow$ a new in-place sorting algorithm in $\mathcal{O}(n \log(n))$

The two operations supported by a priority queue

```
void enqueue(T e, int priority);
T dequeue();
```

This material is explained in some details in last week's lab!

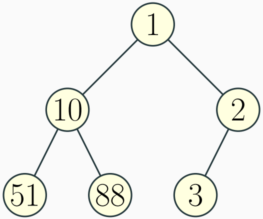
- The last task was marked as challenge because it's about trees and we had not covered that last week
- But now you should try to do it!

What's a heap?

Definition

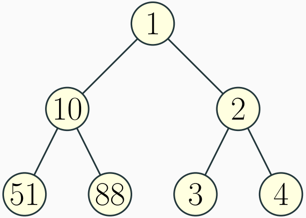
A min-heap is a binary tree such that

- The label of every node is smaller than its children's
- All of its levels are full, except possibly the last
- The last level is completely filled left-to-right

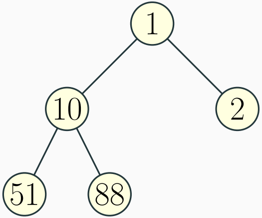


(for priority queues: numbers are priorities + extra label type `T` in nodes)

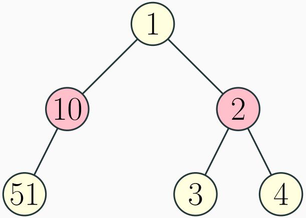
Examples/counter-examples



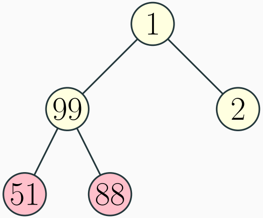
↑ valid heap



↑ valid heap

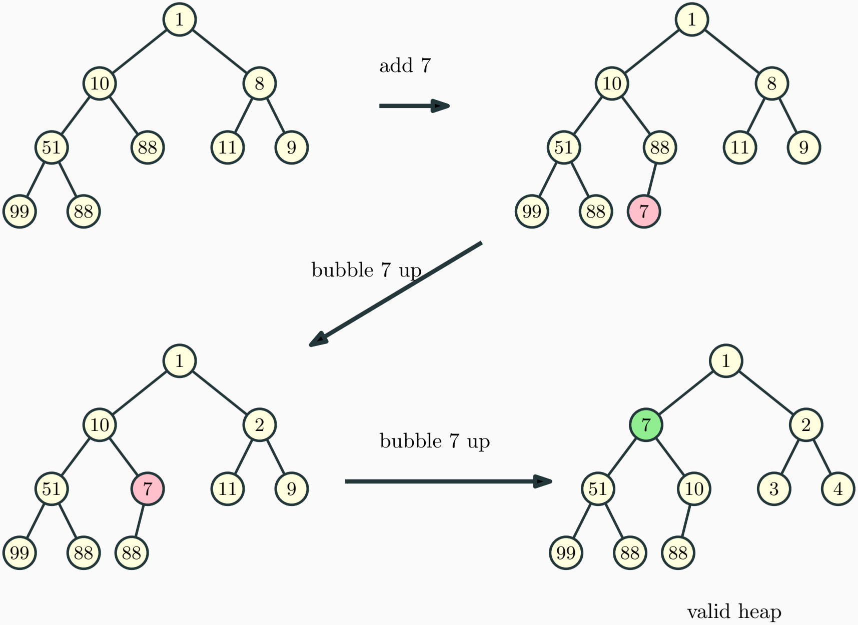


↑ not a heap (unbalanced)

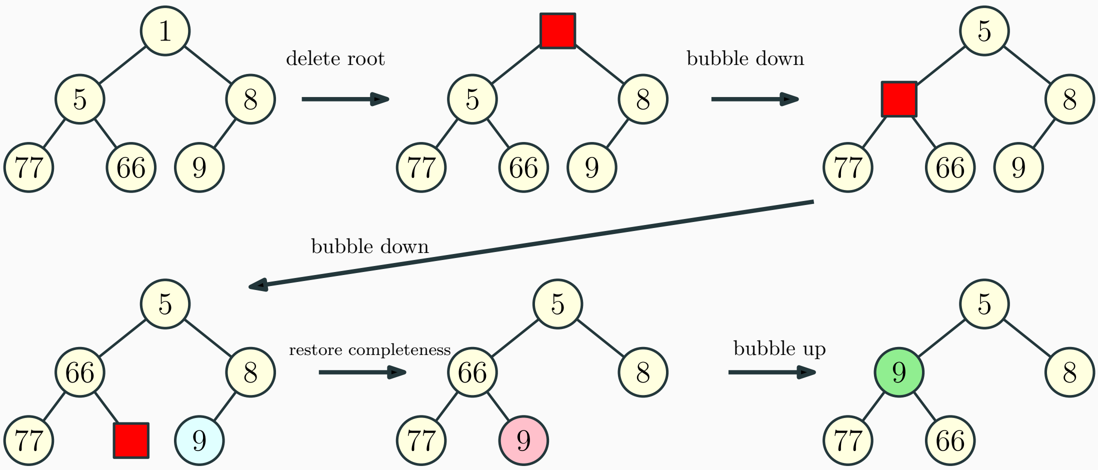


↑ not a heap ($99 > 88$)

Inserting a new element and repairing in $\mathcal{O}(\log(n))$



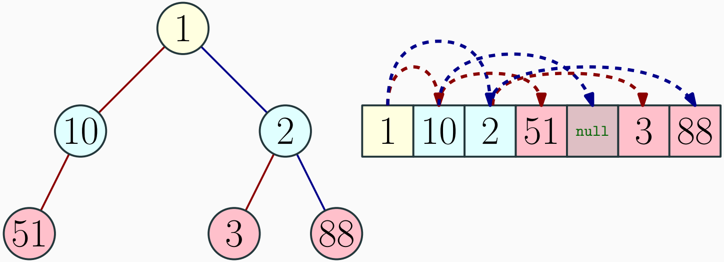
Deleting the root and repairing in $\mathcal{O}(\log(n))$



↑ valid heap

Representing trees as arrays

While the shape of a tree is good ot keep in mind, when they are of bounded arity and close to complete, it might be better to represent them as arrays



- Fast access due to $\mathcal{O}(1)$ lookup in arrays
- Downsides: *potentially* **wasting memory** and bounding a priori arities

(absent nodes = cells filled with **`null`**)

For heaps: that's a good representation!

Heap sort

The algorithm

- start with an empty heap
- insert all the elements in the collection you want sorted

$$\sum_{i=1}^{n} K \log(i) + K' = \mathcal{O}(n \log(n))$$

- insert the value of the root at the back of your output and delete the root

$$\sum_{i=1}^{n} K'' \log(i) + K''' = \mathcal{O}(n \log(n))$$

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  $\sum_{i=1}^{n} K'' \log(i) + K''' = \mathcal{O}(n \log(n))$

- Optimal asymptotic complexity for a comparison-based sort!
- Can be done *in-place* in an array wiht minor adjustement

  $\mathcal{O}(n)$ space complexity

Quick recap on sorting algorithm over arrays (1/2)

Bubble sort

- $\mathcal{O}(n^2)$
- In-place

Quick sort

- $\mathcal{O}(n^2)$, $\mathcal{O}(n\log(n))$ on average with randomized pivot
- Easily done in-place for arrays
- $\mathcal{O}(n\log(n))$ with a smart pivot, but this breaks the in-place aspect of the algo.

Quick recap on sorting algorithm over arrays (1/2)

Merge sort

- $\mathcal{O}(n\log(n))$, good for parallelization
- Not in-place for arrays
- A *stable* sort (does not disturb elements that are "equal")

Heap sort

- $\mathcal{O}(n\log(n))$
- In-place!

CountSort

- Not a comparison-based sort, can run in linear time **if working with numbers in a restricted range**.

That's all for today

See you in the lab to practice working with trees!

Next time we will introduces **graphs**.